

Linear models for thin plates of polymer gels.

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Abstract

Within the linearized three-dimensional theory of polymer gels, we consider a sequence of problems formulated on a family of cylindrical domains whose height tends to zero. We assume that the fluid pressure is controlled at the top and bottom faces of the cylinder, and we consider two different scaling regimes for the diffusivity tensor. Through asymptotic-analysis techniques we obtain two plate models where the transverse displacement is governed by a plate equation with an extra contribution from the fluid pressure. In the limit obtained within the first scaling regime the fluid pressure is affine across the thickness and hence it is determined by its instantaneous trace on the top and bottom faces. In the second model, instead, the value of the fluid pressure is governed by a three-dimensional diffusion equation.

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1 Introduction

A polymer gel is a network of cross-linked polymeric chains permeated by a diffusing fluid. Temporal changes of fluid content bring about local swelling or de-swelling of the gel [14], which in turn may determine substantial changes of shape. Two key properties affecting the amount of swelling are the compliance of the polymeric

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network and the affinity between the polymer and the fluid. Since these properties can be finely tuned during gel synthesis, understanding how they affect shape changes is crucial to controlling and harnessing shape changes.

Swelling-induced shape changes can be particularly dramatic in thin bodies, such as plates or rods [18]. For example, the experiments reported in [25] feature elastic instabilities during swelling of composite polymeric plates. The paper [19] investigates experimentally-observed time dependent bending of a thin plate of polymer gel during the transient that follows the deposition of a solvent droplet on one of its faces.

Mechanical theories of polymer gels and, more generally, of strained solids through which diffusion of a fluid takes place [1, 20, 15, 10] have been elaborated drawing from the work of Gibbs concerning the role of chemical potential to describe the interaction between a solid and a fluid phase [17] and from the work of Biot on poroelastic media [5, 4]. In these theories, a polymeric gel is modeled as a single continuum where diffusion of a chemical species is driven by the gradient of its chemical potential. Constitutive equations are obtained from a dissipation principle that takes explicitly into account the energetic flux associated to the motion of the fluid relative to the solid. A parallel line of research [6, 36, 7] models the diffusing fluid and the strained solid as two superposed and interacting continua, whose governing equations result from the application of Truesdell's theory of mixtures [38]. It would be desirable to have a comparison of the two approaches, showing to what extent they are equivalent. A comparison in the setting of poroelasticity can be found in [8].

In this paper we restrict our considerations to physical situations where the evolution of the polymer gel involves small departures from an equilibrium state. Following [22] we perform a formal linearization procedure of the mechanical equations governing motion and diffusion in a polymer gel proposed in [15, 10]. We then arrive at the following constitutive equations:

$$\mathbf{S} = \mathbb{C}\mathbf{E}\mathbf{u} - p\mathbf{I}, \quad \mathbf{h} = -\nu\mathbf{M}\nabla p, \quad (1)$$

which govern the stress increment \mathbf{S} with respect to the reference state and the flux of diffusant \mathbf{h} . Here, \mathbb{C} is a fourth-order elasticity tensor acting on the linear strain $\mathbf{E}\mathbf{u}$; ν is the molar volume of the fluid, \mathbf{M} is the mobility tensor; the pressure p entering Fick's law (1)₂ is proportional to the fluid's chemical potential and plays the role of Lagrange multiplier associated to the incompressibility constraint

$$\text{tr}(\mathbf{E}\mathbf{u}) = \nu\gamma \quad (2)$$

between the the increment γ of the fluid concentration and the local volume change, as measured by the divergence $\text{div } \mathbf{u} = \text{tr } \mathbf{E}\mathbf{u}$ of the displacement field \mathbf{u} . The

constitutive equations (1) are combined with the balance equations:

$$\operatorname{div} \mathbf{S} + \mathbf{f} = \mathbf{0}, \quad \dot{\gamma} + \operatorname{div} \mathbf{h} = 0 \quad (3)$$

to obtain an evolution system governing displacement \mathbf{u} and pressure p . In problems concerning deformations coupled with diffusion, inertial forces are typically neglected. Then, on identifying \mathbf{f} with a prescribed body force of dead-load type, and on ruling out the unknowns \mathbf{S} and \mathbf{h} through their constitutive equations, we arrive at the following system in the unknowns \mathbf{u} and p :

$$\operatorname{div} \mathbb{C} \mathbf{E} \mathbf{u} - \nabla p = -\mathbf{f}, \quad (4a)$$

$$\operatorname{div} \dot{\mathbf{u}} - \operatorname{div} \mathbf{K} \nabla p = 0, \quad (4b)$$

where $\mathbf{K} = \nu^2 \mathbf{M}$ is the diffusivity tensor.

It is worth noticing that the constitutive equation $(1)_1$ can be formally obtained as a specialization of the equation governing the stress of a saturated linear poroelastic medium [13, Eq. 55] in the case when the effective stress coefficient is equal to one. Moreover, the incompressibility constraint (2) can be interpreted as the limit of the response equation for the pore fluid [13, Eq. 59] in the limit when the Biot modulus tends to infinity. In fact, to a certain extent, the equations of polymer gels can be considered as a specialization of the theory of saturated poroelastic media: Biot himself remarks in [4] that his “pressure function” applies to “[...] *a broader context where it plays the role of “chemical potential”, whose applications are “[...] not restricted to the presence of actual pores. The fluid may be in solution with the solid, or may be adsorbed. Such phenomena are usually associated with the concept of capillarity or osmotic pressures.*” It is also worth noticing that the linear system we consider can be also considered as a special case of the equations of linear thermoelasticity [9, Eq. 7.24] in the case when heat capacity vanishes and the stress–temperature tensor is equal to the identity. Such analogy has been explicitly reckoned, for instance, in [3] and in [19].

These observations put into perspective a number of theories concerning poroelastic and thermoelastic plates as relevant in the context we are considering. An earlier combination of poroelasticity with a structural theory was addressed by Biot, who considers in [3] the effect of fluid flow within a poroelastic slab under compression, and who shows that creep buckling takes place when the compression is above a “lower” critical value, and that if the load is increased then the rate of lateral deflection increases until it becomes infinite at an “upper” critical value. In [37] Taber proposes a theory describing the coupling between diffusion and bending in a poroelastic plate, based on the Kirchhoff–Love kinematics, on the plane-strain

assumption, and on the assumption that diffusion of fluid takes place only in the direction orthogonal to the mid-plane of the plate.

A theory that specifically addresses thin polymer gels coupled with fluid diffusion has more recently been proposed in [22] concerning a planar beam whereby motion and diffusion are restricted to two dimensions. In that paper, the constitutive equations governing the relation between bending moment and curvature are obtained from the standard kinematic assumptions that the cross section of the beam remain straight, and that the stress along the transverse direction vanishes. Within these assumptions, two different models have been proposed: a two-dimensional diffusion model which allows the fluid to diffuse both in the axial and in the transverse direction of the beam; a one-dimensional diffusion model which allows fluid diffusion only in the transverse direction, as in the above-mentioned paper by Taber. Comparison between the two models shows that the one dimensional model is still adequate to capture the response of the beam to pressure variations. More recently, the equations of a poroelastic plate under large strain have been derived using the virtual-powers approach combined with an enriched kinematics [23].

All the above-mentioned derivations make use of *ad hoc* hypotheses to obtain a structural model. There is however a number of papers dealing with poroelasticity and thermoelasticity which make use of rigorous asymptotic analysis to obtain structural (plate or beam) models, in the spirit of [12]: one considers a *generating family* of initial-boundary value depending on a vanishing parameter ε , designed in a way to capture the thinness of the body under scrutiny (be it a plate or a beam), and studies the asymptotic behavior of the solutions of these problems as ε tends to null. One may naively expect that such a procedure, being rigorous, produces a definite answer concerning what structural model better captures thinness. However, this is not the case: as pointed out in [29], the outcome of the asymptotic method depends crucially on the *choice* of the generating family, and different choices lead to different structural models. For example, there is more than one choice that can lead to the Reissner–Mindlin plate model [31, 30, 26, 34], to models used for plate buckling [32], plates with residual stress [27, 28] or to the Timoshenko beam theory [35, 16].

The generating families are usually obtained by writing the relevant equations (in our case, (1)–(3)) on a collection of cylinders

$$\Omega_\varepsilon = \omega \times \left(-\varepsilon \frac{h}{2}, +\varepsilon \frac{h}{2} \right), \quad \omega \subset \mathbb{R}^2, \quad (5)$$

which shrink to the planar domain $\omega \times \{0\}$ as ε tends to null. The coefficients in these equations, as well as the boundary conditions, can possibly depend on the parameter ε . Most of these theories are essentially a variant of the Kirchhoff–Love

plate equation or Euler–Bernoulli beam equation with an extra contribution to the bending moment coming from pressure inhomogeneities across the thickness. This contribution evolves in time as determined by the flow of fluid or thermal conduction within the plate and is itself affected by bending.

In the simplest setting of three-dimensional linear elasticity, the most popular approach to the derivation of plate equations by asymptotic analysis is to keep the elasticity tensor \mathbb{C} independent of ε , and replace the displacement field \mathbf{u}_ε with the scaled displacement \mathbf{u}^ε whose components are defined as (*cf.* Eq. 74 in Sec. 5):

$$u_\alpha^\varepsilon = \frac{1}{\varepsilon}(\mathbf{u}_\varepsilon)_\alpha, \quad u_3^\varepsilon = (\mathbf{u}_\varepsilon)_3. \quad (6)$$

It appears natural to replicate this pattern when mechanical effects are coupled with diffusion, and a simple calculation (see the chain of argument that leads from Eq. 90 to Eq. 96) shows that if one wants the pressure p to appear in the limit equation, then the original unknown p_ε should be rescaled by introducing the new unknown

$$p^\varepsilon = \frac{p_\varepsilon}{\varepsilon}. \quad (7)$$

These scaling are essentially common to all asymptotic derivations we were able to find in the literature. However, there are still many degrees of freedom in the choice of the generating family of problems. This explains why so many different limit theories can be obtained. For example, the derivation of theories of thermoelastic plates presented in [21], where temperature plays the role of pressure, makes use of Robin-type conditions on the top and bottom faces of the cylinder and considers three different scalings, which result in different limit problems. The paper [24], which is concerned with poroelastic plates, considers the regime where the diffusivity tensor \mathbf{K} scales as ε^2 . In this case, two systems of partial differential equations are obtained: the first is an elliptic system formulated on the 2D domain ω , which governs the in-plane components of displacement and the average pressure across the thickness; the second is a system composed of the classical plate equation formulated on ω , coupled with a particular diffusion equation formulated on the domain Ω_1 whose peculiarity is that diffusion can take place only in the transverse direction. The result is essentially the same as the 1D stress-diffusion model discussed in [22].

In this paper we consider material symmetry of monoclinic type with respect to the plane containing ω , and we build two different generating families. For the first family, the diffusivity tensor \mathbf{K} is independent of ε , so that the only datum that depends on ε is the thickness of the domain Ω_ε . We show in Section 5 that this family produces a two dimensional model where the value of the pressure across

the thickness is the affine interpolation of the values of the pressure at the top and bottom face, as in theories describing thermal bending. For the second family we consider, the planar components of the diffusivity tensor are constant, while the transverse component scales as ε^2 . As we show in Section 6, this family generates the diffusive model proposed in [22].

2 A nonlinear system of evolution equations

In this section, we put together the system of partial differential equations that arises in mechanical theories that describe diffusion of a solvent through a finitely-strained solid, and we linearize the resulting equations to arrive at an initial-boundary value problem that is the object of our asymptotic analysis. For the sake of conciseness, we limit ourselves to presenting the key ingredients and we refer the reader eager for further details to the many presentations available in the literature, which we have listed in the Introduction. Moreover, we leave for the next section the specification of the relevant boundary and initial conditions.

As a start we recall, that for Ω the region occupied by the body in its reference configuration, and for $I = [0, T]$ the time interval of interest, the unknown fields for the theory in question are: the *deformation* f , the *Piola stress* \mathbf{S} , the *referential concentration of solvent* c (a positive-valued scalar field), the *referential solvent flux* \mathbf{h} , and the *solvent chemical potential* μ .

For \mathbf{f} the dead body-force field, the aforementioned fields are required to satisfy the force- and the mass-balance:

$$\operatorname{div} \mathbf{S} + \mathbf{f} = \mathbf{0}, \quad (8a)$$

$$\dot{c} + \operatorname{div} \mathbf{h} = 0, \quad (8b)$$

as well as a set of constitutive equations consistent with the dissipation inequality

$$\dot{\psi} \leq \mathbf{S} \cdot \dot{\mathbf{F}} + \mu \dot{c} - \mathbf{h} \cdot \nabla \mu, \quad (9)$$

where

$$\mathbf{F} = \nabla f \quad (10)$$

is the deformation gradient.

For applications to gels, it is usually assumed that both the polymer and the solvent be incompressible. This assumption does not imply that the admissible deformations of the gel as a whole are isochoric: volume changes, as measured by the determinant $\det \mathbf{F}$ of the deformation gradient, may take place due to changes

of the relative volume proportions of polymer and solvent. However, these changes must comply with the *incompressibility constraint*:

$$\det \mathbf{F} = 1 + \nu(c - \mathring{c}), \quad (11)$$

where \mathring{c} is the number of solvent molecules per unit referential volume and ν is molecular volume. For $\mathbf{F}^\star = \det \mathbf{F} \mathbf{F}^{-T}$ the cofactor of \mathbf{F} , we have from (11) that the rate of change of concentration is related to the rate of deformation gradient by $\dot{c} = \nu^{-1} \mathbf{F}^\star \cdot \dot{\mathbf{F}}$. As a result, (9) yields

$$\dot{\psi} \leq \left(\mathbf{S} + \frac{\mu}{\nu} \mathbf{F}^\star \right) \cdot \dot{\mathbf{F}} - \mathbf{h} \cdot \nabla \mu. \quad (12)$$

By comparing (9) with (21) we see that an important consequence of the incompressibility constraint (11) is that the Piola stress \mathbf{S} expend power in concomitance with the chemical potential μ , and hence only a combination of these fields, namely $\mathbf{S} + \mu/\nu \mathbf{F}^\star$, can be constitutively prescribed. Thus, in view of (21), we adopt the following constitutive equations:

$$\mathbf{S} = \psi'(\mathbf{F}) - \frac{\mu}{\nu} \mathbf{F}^\star \quad (13a)$$

$$\mathbf{h} = -\mathbf{M}(c) \nabla \mu, \quad (13b)$$

where $\mathbf{M}(c)$ is a concentration-dependent, positive definite mobility tensor. Viewing (13a) as a constitutive equation for \mathbf{S} leaves the chemical potential not specified constitutively. In fact, within this theory, the chemical potential cannot be determined only by solving the complete problem. The mechanical interpretation of μ comes through the decomposition

$$\mu = \mu^0 + \nu \sigma, \quad (14)$$

where μ^0 is the *chemical potential of the pure solvent*, and σ is the *solvent pressure*.

When put together, the balance equations (8), the compatibility condition (10), the incompressibility constraint (11), and the constitutive equations (13) form a system of partial differential equations in the unknowns $(f, \mathbf{F}, \mathbf{S}, c, \mu, \mathbf{h})$.

3 Linearization of the evolution equations

Our next task is to perform a formal linearization of this system about a suitable reference state. As a first step, we take as reference state an equilibrium solution of the aforementioned system, where the body is undeformed, so that $\mathring{f}(x) = x$ and

$\mathring{\mathbf{F}} = \mathbf{I}$, and both stress and solvent flux vanish. We stick to the convention of marking by a superimposed ring all fields that pertain to the reference state. Accordingly, by (13) we write

$$\mathring{\mathbf{S}} = \psi'(\mathbf{I}) - \frac{\mathring{\mu}}{\nu} \mathbf{I} = \mathbf{0}, \quad (15a)$$

$$\mathring{\mathbf{h}} = \mathbf{M}(\mathring{c}) \nabla \mathring{\mu} = \mathbf{0}. \quad (15b)$$

In particular, in the reference state the solvent pressure is

$$\mathring{\sigma} = \frac{\mathring{\mu} - \mu^0}{\nu}. \quad (16)$$

As our second step we consider small departures from the reference state and we write

$$f(x) = x + \mathbf{u}(x), \quad \mathbf{F} = \mathbf{I} + \mathbf{H}, \quad c = \mathring{c} + \gamma, \quad \mu = \mathring{\mu} + \nu p, \quad (17)$$

where \mathbf{u} is the *displacement*, $\mathbf{H} = \nabla \mathbf{u}$ is the *displacement gradient*, γ is the *concentration increment*, and (cf. (14) and (16))

$$p = \sigma - \mathring{\sigma}, \quad (18)$$

is the *solvent pressure increment*.

In the following we assume that the loads and the boundary conditions are such that $|\mathbf{H}|$ is small and the increment of pressure solvent p is also of the same order. From the incompressibility constraint (11), it follows that also the concentration increment γ is of order $|\mathbf{H}|$.

The following linearization formula:

$$\mathbf{F}^* \cong \mathbf{I} + (\mathbf{I} \cdot \mathbf{H}) \mathbf{I} - \mathbf{H}^T, \quad (19)$$

when applied to (13a) allows us to write:

$$\mathbf{S} \cong \psi'(\mathbf{I}) + \psi''(\mathbf{I})[\mathbf{H}] - \frac{\mathring{\mu}}{\nu} (\mathbf{I} + (\mathbf{I} \cdot \mathbf{H}) \mathbf{I} - \mathbf{H}^T) - p \mathbf{I},$$

where the symbol \cong indicates that the equality holds up to infinitesimals of first order. By making use of the representation formula

$$\psi''(\mathbf{I})[\mathbf{H}] = \text{sym}(\psi''(\mathbf{I})[\mathbf{E}]) + \mathbf{H} \psi'(\mathbf{I}) - \text{sym}(\mathbf{E} \psi'(\mathbf{I})), \quad \mathbf{E} = \text{sym} \mathbf{H}, \quad (20)$$

which follows from frame indifference, and recalling (15a) we find that

$$\begin{aligned}
\mathbf{S} &\cong \psi'(\mathbf{I}) - \frac{\dot{\mu}}{\nu} \mathbf{I} + \text{sym}(\psi''(\mathbf{I})[\mathbf{E}]) + \mathbf{H}\psi'(\mathbf{I}) - \text{sym}(\mathbf{E}\psi'(\mathbf{I})) \\
&\quad - \frac{\dot{\mu}}{\nu}((\mathbf{I} \cdot \mathbf{H})\mathbf{I} - \mathbf{H}^T) - p\mathbf{I}, \\
&= \text{sym}(\psi''(\mathbf{I})[\mathbf{E}]) + \frac{\dot{\mu}}{\nu} \mathbf{H} - \text{sym}(\mathbf{E}\psi'(\mathbf{I})) - \frac{\dot{\mu}}{\nu}((\mathbf{I} \cdot \mathbf{H})\mathbf{I} - \mathbf{H}^T) - p\mathbf{I}, \\
&= \text{sym}(\psi''(\mathbf{I})[\mathbf{E}]) + \frac{\dot{\mu}}{\nu}(\mathbf{E} - (\mathbf{I} \cdot \mathbf{E})\mathbf{I}) - p\mathbf{I}.
\end{aligned} \tag{21}$$

On introducing the fourth-order tensor \mathbb{C} defined by

$$\mathbb{C}[\mathbf{E}] = \text{sym}(\psi''(\mathbf{I})[\mathbf{E}]) + \frac{\dot{\mu}}{\nu}(\mathbf{E} - (\mathbf{I} \cdot \mathbf{E})\mathbf{I}). \tag{22}$$

we can rewrite (21) as

$$\mathbf{S} \cong \mathbb{C}[\mathbf{E}] - p\mathbf{I}. \tag{23}$$

Next, we notice that (15b), together with the positive definiteness of the mobility tensor, implies

$$\nabla \dot{\mu} = 0, \tag{24}$$

hence, when we linearize(21) we obtain,

$$\begin{aligned}
\mathbf{h} &\cong -\mathbf{M}(\dot{\mathbf{c}})\nabla \dot{\mu} - \gamma \mathbf{M}'(\dot{\mathbf{c}})\nabla \dot{\mu} - \nu \mathbf{M}(\dot{\mathbf{c}})\nabla p \\
&= -\nu \mathbf{M}(\dot{\mathbf{c}})\nabla p.
\end{aligned} \tag{25}$$

By putting together the balance equations (8), the linearized constitutive equations (23) and (25), and the linearization of the incompressibility constraint (11), we obtain the following *incremental system*:

$$\begin{aligned}
\text{div } \mathbf{S} + \mathbf{b} &= \mathbf{0} \\
\mathbf{S} &\cong \mathbb{C}[\mathbf{E}\mathbf{u}] - p\mathbf{I} \\
\dot{\gamma} &\cong \nu \text{div}(\mathbf{M}(\dot{\mathbf{c}})\nabla p) \quad \text{in } \Omega, \\
\mathbf{I} \cdot \mathbf{E}\mathbf{u} &\cong 1 + \nu\gamma \\
\mathbf{E}\mathbf{u} &= \text{sym} \nabla \mathbf{u}
\end{aligned} \tag{26}$$

where we wrote explicitly the dependence on the displacement \mathbf{u} of the strain $\mathbf{E}\mathbf{u} = \text{sym} \nabla \mathbf{u}$. Our final step consists in the elimination of the unknowns \mathbf{S} and γ from (26), a step that leads to the following system in the unknowns \mathbf{u} and p :

$$\text{div } \mathbb{C}[\mathbf{E}\mathbf{u}] - \nabla p = -\mathbf{f}, \tag{27a}$$

$$\text{div } \dot{\mathbf{u}} = \nu^2 \text{div} \mathbf{M}(\dot{\mathbf{c}})\nabla p. \tag{27b}$$

From now on we set $\mathbf{K} = \nu^2 \mathbf{M}(\dot{\mathbf{c}})$.

4 The existence and uniqueness of the solution of the three-dimensional problem

The linearization procedure carried out in the previous section leads to the following system of equations

$$\operatorname{div} \mathbb{C} \mathbf{E} \mathbf{u} - \nabla p = -\mathbf{f}, \quad (28a)$$

$$\operatorname{div} \dot{\mathbf{u}} - \operatorname{div} \mathbf{K} \nabla p = 0, \quad (28b)$$

formulated on a space–time region $\Omega \times (0, T)$, where Ω is a domain of \mathbb{R}^3 .

For the application to plates we have in mind, the following combination of boundary conditions is relevant:

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_{u,D}, \quad (\mathbb{C} \mathbf{E} \mathbf{u}(t)) \mathbf{n} - p(t) \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{u,N}, \quad (29a)$$

$$p(t) = p_a(t) \text{ on } \Gamma_{p,D}, \quad -\mathbf{K} \nabla p(t) \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{p,N}, \quad (29b)$$

where $\partial\Omega = \Gamma_{u,D} \cup \Gamma_{u,N}$ and $\partial\Omega = \Gamma_{p,D} \cup \Gamma_{p,N}$ are partitions of $\partial\Omega$. Here p_a is a pressure increment applied $\Gamma_{p,D}$. The formulation of an initial–boundary value problems is arrived at by imposing an initial condition for the pressure:

$$p(0) = p_0, \quad (30)$$

with p_0 compatible with (75b).

We dedicate the rest of this section to proving the existence and the uniqueness of solutions for Problem (28)–(30) in the case when the applied pressure increment vanishes:

$$p_a = 0. \quad (31)$$

We leave to a remark at the end of this section the illustration of how to treat the more general case where the applied pressure p_a does not vanish.

Notation. We denote by $\mathbb{R}_{\text{Sym}}^{3 \times 3}$ the vector space of symmetric 3×3 matrices with real entries. We introduce the function spaces

$$H_{u,D}^1(\Omega; \mathbb{R}^3) = \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{u,D}\}$$

and

$$H_{p,D}^1(\Omega) = \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_{p,D}\},$$

respectively, for the displacement \mathbf{u} and the pressure p . For B a Banach space, we denote by $L^p(0, T; B)$ the standard Lebesgue spaces of Bochner-integrable functions

defined on the time interval $(0, T)$ and taking values in B . Likewise, we denote by $H^1(0, T; B)$ the corresponding Sobolev space of functions whose derivative with respect to time is in $L^2(0, T; B)$. For typographical convenience, when denoting the Lebesgue or Sobolev norm of functions defined on Ω and taking values in \mathbb{R}^N , we omit the specification of the domain and the codomain. For example, we write $\|\cdot\|_{L^q}$ and $\|\cdot\|_{L^p(0, T; L^q)}$ in place of $\|\cdot\|_{L^q(\Omega; \mathbb{R}^n)}$ and $\|\cdot\|_{L^p(0, T; L^q(\Omega; \mathbb{R}^n))}$, respectively.

Assumptions. We shall make the following assumptions concerning the data:

$$\mathcal{H}^2(\Gamma_{u,D}) > 0, \mathcal{H}^2(\Gamma_{p,D}) > 0, \quad (32a)$$

$$p_0 \in H_{p,D}^1(\Omega), \quad (32b)$$

$$\mathbf{f} \in H^1(0, T; L^2(\Omega; \mathbb{R}^3)), \quad (32c)$$

$$\mathbb{C}_{ijkl} \in L^\infty(\Omega), \quad (32d)$$

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}, \quad (32e)$$

$$\exists c_{\mathbb{C}} > 0 : \mathbb{C}\mathbf{A} \cdot \mathbf{A} \geq c_{\mathbb{C}}|\mathbf{A}|^2 \quad \forall \mathbf{A} \in \mathbb{R}_{\text{Sym}}^{3 \times 3} \quad \text{a.e. in } \Omega, \quad (32f)$$

$$\mathbf{K} \in L^\infty(\Omega; \mathbb{R}_{\text{Sym}}^{3 \times 3}), \quad (32g)$$

$$\exists c_{\mathbf{K}} > 0 : \mathbf{K}\mathbf{a} \cdot \mathbf{a} \geq c_{\mathbf{K}}|\mathbf{a}|^2 \quad \forall \mathbf{a} \in \mathbb{R}^3 \quad \text{a.e. in } \Omega. \quad (32h)$$

Theorem 1. *Under assumptions (32), the initial-boundary-value problem (28)-(30) has a unique solution in the following weak sense:*

$$\left\{ \begin{array}{l} \mathbf{u} \in H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ p \in L^\infty(0, T; H_{p,D}^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \text{with} \quad p(0) = p_0, \\ \int_{\Omega} (\mathbb{C}\mathbf{E}\mathbf{u}(t) \cdot \mathbf{E}\mathbf{v} - p(t) \operatorname{div} \mathbf{v}) \, dx = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} \, dx \\ \quad \forall \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3), \forall t \in [0, T], \\ \int_{\Omega} (\operatorname{div} \dot{\mathbf{u}}(t) q + \mathbf{K} \nabla p(t) \cdot \nabla q) \, dx = 0 \quad \forall q \in H_{p,D}^1(\Omega), \text{ for a.e. } t \in (0, T). \end{array} \right. \quad (33)$$

We perform the proof of existence in several steps.

4.1 Time discretization.

Our first step is the construction of a sequence of approximate solutions to (28)-(29). Our approximation scheme is based on a time discretization with time step $\tau = T/n$, with n an integer that we shall eventually let to $+\infty$.

With a view towards setting up our iterative scheme, we begin by approximating the bulk datum \mathbf{f} , with the sequence

$$\mathbf{f}_\tau^k = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} \mathbf{f}(t) dt \in L^2(\Omega; \mathbb{R}^3) \quad \text{for } k = 1 \dots n, \quad (34)$$

and we set

$$\mathbf{f}_\tau^0 = \mathbf{f}(0),$$

where $\mathbf{f}(0)$, understood in the sense of traces, is well defined thanks to the time regularity of the forcing term \mathbf{f} stipulated in Assumption (32c).

Next, we define the initial displacement \mathbf{u}_0 as the unique solution of the variational problem:

$$\begin{cases} \mathbf{u}_0 \in H_{u,D}^1(\Omega; \mathbb{R}^3), \\ \int_{\Omega} (\mathbb{C} \mathbf{E} \mathbf{u}_0 \cdot \mathbf{E} \mathbf{v} - p_0 \operatorname{div} \mathbf{v}) dx = \int_{\Omega} \mathbf{f}_\tau^0 \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3). \end{cases} \quad (35)$$

The existence and the uniqueness of the solution of (35) follows immediately from the Lax–Milgram lemma. Now, for

$$\mathbf{u}_\tau^0 = \mathbf{u}_0, \quad p_\tau^0 = p_0 \quad (36)$$

we formulate recursively, starting from $k = 1$ up to $k = n$, a sequence of problems consisting in the system of partial differential equations:

$$\operatorname{div} \mathbb{C} \mathbf{E} \mathbf{u}_\tau^k - \nabla p_\tau^k = -\mathbf{f}_\tau^k, \quad (37a)$$

$$\operatorname{div} \left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right) - \operatorname{div} \mathbf{K} \nabla p_\tau^k = 0, \quad (37b)$$

in the unknowns \mathbf{u}_τ^k and p_τ^k , supplemented by the boundary conditions:

$$\mathbf{u}_\tau^k = 0 \text{ on } \Gamma_{u,D}, \quad \mathbb{C}(\mathbf{E} \mathbf{u}_\tau^k) \mathbf{n} - p_\tau^k \mathbf{n} = 0 \text{ on } \Gamma_{u,N}, \quad (38a)$$

$$p_\tau^k = 0 \text{ on } \Gamma_{p,D}, \quad \mathbf{K} \nabla p_\tau^k \cdot \mathbf{n} = 0 \text{ on } \Gamma_{p,N}. \quad (38b)$$

Our next step is to establish the existence of a weak solution to (37)-(38) for all k .

Proposition 1. *For $k \in \{1, \dots, n\}$, let $\mathbf{u}_\tau^{k-1} \in H_{u,D}^1(\Omega; \mathbb{R}^3)$ be given. Then the*

boundary-value problem (37)-(38) has a unique solution in the following sense:

$$\begin{cases} \mathbf{u}_\tau^k \in H_{u,D}^1(\Omega; \mathbb{R}^3), \\ p_\tau^k \in H_{p,D}^1(\Omega), \\ \int_\Omega (\mathbb{C}\mathbf{E}\mathbf{u}_\tau^k \cdot \mathbf{E}\mathbf{v} - p_\tau^k \operatorname{div} \mathbf{v}) \, dx = \int_\Omega \mathbf{f}_\tau^k \cdot \mathbf{v} \, dx \quad \forall \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3), \\ \int_\Omega (\operatorname{div} \mathbf{u}_\tau^k q + \tau \mathbf{K} \nabla p_\tau^k \cdot \nabla q) \, dx = \int_\Omega \operatorname{div} \mathbf{u}_\tau^{k-1} q \, dx \quad \forall q \in H_{p,D}^1(\Omega). \end{cases} \quad (39)$$

Proof. We introduce the Hilbert space $H = H_{u,D}^1(\Omega; \mathbb{R}^3) \times H_{p,D}^1(\Omega)$ equipped with the scalar product

$$((\mathbf{u}, p), (\mathbf{v}, q))_H = (\mathbf{u}, \mathbf{v})_{H^1} + (p, q)_{H^1}. \quad (40)$$

We define the bilinear form $a_\tau : H \times H \rightarrow \mathbb{R}$ and the linear functional $\ell_\tau^k : H \rightarrow \mathbb{R}$ defined by

$$a_\tau((\mathbf{u}, p), (\mathbf{v}, q)) = \int_\Omega (q \operatorname{div} \mathbf{u} + \tau \mathbf{K} \nabla p \cdot \nabla q + \mathbb{C}\mathbf{E}\mathbf{u} \cdot \mathbf{E}\mathbf{v} - p \operatorname{div} \mathbf{v}) \, dx, \quad (41)$$

and

$$\ell_\tau^k(\mathbf{v}, q) = \int_\Omega (\mathbf{f}_\tau^k \cdot \mathbf{v} + \operatorname{div} \mathbf{u}_\tau^{k-1} q) \, dx. \quad (42)$$

Problem (39) can then be rewritten as follows:

$$a_\tau((\mathbf{u}_\tau^k, p_\tau^k), (\mathbf{v}, q)) = \ell_\tau^k(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in H. \quad (43)$$

Since

$$a_\tau((\mathbf{u}, p), (\mathbf{v}, q)) \leq C \|(\mathbf{u}, p)\|_H \|(\mathbf{v}, q)\|_H \quad \forall (\mathbf{u}, p), (\mathbf{v}, q) \in H, \quad (44)$$

the bilinear form a_τ is continuous. Moreover, by Korn's and Poincaré's inequalities, we have

$$\begin{aligned} a_\tau((\mathbf{u}, p), (\mathbf{u}, p)) &= \int_\Omega (\tau \mathbf{K} \nabla p \cdot \nabla p + \mathbb{C}\mathbf{E}\mathbf{u} \cdot \mathbf{E}\mathbf{u}) \, dx \\ &\geq \tau C_{\mathbf{K}} \|\nabla p\|_{L^2}^2 + C_{\mathbb{C}} \|\mathbf{E}\mathbf{u}\|_{L^2}^2 \\ &\geq \tau C \|(\mathbf{u}, p)\|_H^2 \end{aligned} \quad \forall (\mathbf{u}, p) \in H, \quad (45)$$

hence the bilinear form $a_\tau(\cdot, \cdot)$ is coercive for each τ . The existence of a unique solution to (43) follows from the Lax-Milgram Lemma. \square

4.2 A priori estimates for the time-discrete problem.

We shall make use of the following discrete version of Gronwall's inequality (see [33, Lemma 1.4.2]):

Lemma 1. *Let $\{a_k\}$ and $\{b_k\}$ be non-negative sequences and let $c_0 > 0$. If the sequence $\{y_k\}$ satisfies*

$$y_k \leq c_0 + \sum_{j=1}^{k-1} (a_j + b_j y_j), \quad (46)$$

then

$$y_k \leq \left(c_0 + \sum_{j=1}^{k-1} a_j\right) e^{\sum_{j=1}^{k-1} b_j}. \quad (47)$$

Moreover, we shall make use of the following result.

Lemma 2. *For $k = 1 \dots n$, the interpolants \mathbf{f}_τ^k defined in (34) satisfy the inequalities*

$$\tau \sum_{k=1}^n \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2}^2 \leq C, \quad \|\mathbf{f}_\tau^k\|_{L^2} \leq C,$$

with C a constant that does not depend on τ .

Proof. On exploiting the time-regularity of \mathbf{f} stipulated in Assumption (32c) we deduce the following chain of equalities and inequalities for $k \geq 1$:

$$\begin{aligned} \|\mathbf{f}_\tau^{k+1} - \mathbf{f}_\tau^k\|_{L^2}^2 &= \left\| \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \int_{t-\tau}^t \dot{\mathbf{f}}(s) ds dt \right\|_{L^2}^2 \leq \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \left\| \int_{t-\tau}^t \dot{\mathbf{f}}(s) ds \right\|_{L^2}^2 dt \\ &= \tau \int_{k\tau}^{(k+1)\tau} \left\| \frac{1}{\tau} \int_{t-\tau}^t \dot{\mathbf{f}}(s) ds \right\|_{L^2}^2 dt \leq \tau \int_{(k-1)\tau}^{(k+1)\tau} \|\dot{\mathbf{f}}(s)\|_{L^2}^2 ds. \end{aligned}$$

By the same token, we have

$$\begin{aligned} \|\mathbf{f}_\tau^1 - \mathbf{f}_\tau^0\|_{L^2}^2 &= \left\| \frac{1}{\tau} \int_0^\tau (\mathbf{f}(t) - \mathbf{f}(0)) dt \right\|_{L^2}^2 = \left\| \frac{1}{\tau} \int_0^\tau \int_0^t \dot{\mathbf{f}}(s) ds dt \right\|_{L^2}^2 \\ &\leq \frac{1}{\tau} \int_0^\tau \left\| \int_0^t \dot{\mathbf{f}}(s) ds \right\|_{L^2}^2 dt \\ &\leq \tau \int_0^\tau \|\dot{\mathbf{f}}(s)\|_{L^2}^2 ds. \end{aligned}$$

On combining the above inequalities we have the first inequality of the thesis. The second inequality is a consequence of the first inequality and of the following chain:

$$\begin{aligned}
\|\mathbf{f}_\tau^k\|_{L^2} &\leq \|\mathbf{f}_\tau^0\|_{L^2} + \sum_{j=1}^k \|\mathbf{f}_\tau^j - \mathbf{f}_\tau^{j-1}\|_{L^2} \\
&\leq \|\mathbf{f}_\tau^0\|_{L^2} + \sum_{j=1}^n \|\mathbf{f}_\tau^j - \mathbf{f}_\tau^{j-1}\|_{L^2} \\
&= \|\mathbf{f}_\tau^0\|_{L^2} + \sum_{j=1}^n \tau \left\| \frac{\mathbf{f}_\tau^j - \mathbf{f}_\tau^{j-1}}{\tau} \right\|_{L^2} \\
&\leq \|\mathbf{f}_\tau^0\|_{L^2} + T^{1/2} \left(\sum_{j=1}^n \tau \left\| \frac{\mathbf{f}_\tau^j - \mathbf{f}_\tau^{j-1}}{\tau} \right\|_{L^2}^2 \right)^{1/2}.
\end{aligned}$$

□

The estimates we are going to derive are best written in terms of the following interpolants:

$$\left. \begin{aligned}
\bar{\mathbf{u}}_\tau(t) &= \mathbf{u}_\tau^k, \\
\bar{p}_\tau(t) &= p_\tau^k, \\
\mathbf{u}_\tau(t) &= \mathbf{u}_\tau^{k-1} + (t - (k-1)\tau) \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \\
p_\tau(t) &= p_\tau^{k-1} + (t - (k-1)\tau) \frac{p_\tau^k - p_\tau^{k-1}}{\tau}
\end{aligned} \right\} \begin{aligned} &\text{for } k = 1, \dots, n \\ &\text{and } t \in ((k-1)\tau, k\tau]. \end{aligned} \quad (48)$$

Proposition 2 (Energetic estimates). *The estimates*

$$\|\bar{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^1)} \leq C, \quad (49a)$$

$$\|\bar{p}_\tau\|_{L^2(0,T;H^1)} \leq C, \quad (49b)$$

hold uniformly with respect to τ .

Proof. For $k \in \{1 \dots n\}$ we choose $\mathbf{v} = \mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}$ and $q = p_\tau^k$ as test functions in the weak formulation (39) of the discrete scheme, and we add the resulting equations to obtain:

$$\int_{\Omega} (\mathbb{C} \mathbf{E} \mathbf{u}_\tau^k \cdot \mathbf{E}(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}) + \tau \mathbf{K} \nabla p_\tau^k \cdot \nabla p_\tau^k) \, dx = \int_{\Omega} \mathbf{f}_\tau^k \cdot (\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}) \, dx. \quad (50)$$

The symmetry and the positivity of the elasticity tensor imply the inequality $2\mathbb{C}\mathbf{A} \cdot (\mathbf{A} - \mathbf{B}) \geq \mathbb{C}\mathbf{A} \cdot \mathbf{A} - \mathbb{C}\mathbf{B} \cdot \mathbf{B}$ for every pair of symmetric tensors \mathbf{A} and \mathbf{B} . We therefore arrive at the following inequality:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \mathbb{C}\mathbf{E}\mathbf{u}_{\tau}^k \cdot \mathbf{E}\mathbf{u}_{\tau}^k dx - \frac{1}{2} \int_{\Omega} \mathbb{C}\mathbf{E}\mathbf{u}_{\tau}^{k-1} \cdot \mathbf{E}\mathbf{u}_{\tau}^{k-1} dx + \tau \int_{\Omega} \mathbf{K} \nabla p_{\tau}^k \cdot \nabla p_{\tau}^k dx \\ & \leq \int_{\Omega} \mathbf{f}_{\tau}^k \cdot (\mathbf{u}_{\tau}^k - \mathbf{u}_{\tau}^{k-1}) dx. \end{aligned} \quad (51)$$

Let $1 \leq j \leq n = T/\tau$. From (51), by performing summation of both sides with k running from 1 to j we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \mathbb{C}\mathbf{E}\mathbf{u}_{\tau}^j \cdot \mathbf{E}\mathbf{u}_{\tau}^j dx + \sum_{k=1}^j \tau \int_{\Omega} \mathbf{K} \nabla p_{\tau}^k \cdot \nabla p_{\tau}^k dx \\ & \leq \frac{1}{2} \int_{\Omega} \mathbb{C}\mathbf{E}\mathbf{u}_{\tau}^0 \cdot \mathbf{E}\mathbf{u}_{\tau}^0 dx + \sum_{k=1}^j \int_{\Omega} \mathbf{f}_{\tau}^k \cdot (\mathbf{u}_{\tau}^k - \mathbf{u}_{\tau}^{k-1}) dx. \end{aligned} \quad (52)$$

Next, we have

$$\begin{aligned} & \sum_{k=1}^j \int_{\Omega} \mathbf{f}_{\tau}^k \cdot (\mathbf{u}_{\tau}^k - \mathbf{u}_{\tau}^{k-1}) dx \\ & = - \sum_{k=1}^{j-1} \int_{\Omega} (\mathbf{f}_{\tau}^{k+1} - \mathbf{f}_{\tau}^k) \cdot \mathbf{u}_{\tau}^k dx + \int_{\Omega} (\mathbf{f}_{\tau}^j \cdot \mathbf{u}_{\tau}^j - \mathbf{f}_{\tau}^1 \mathbf{u}_{\tau}^0) dx, \\ & \leq \sum_{k=1}^{j-1} \left(\frac{1}{2\tau} \|\mathbf{f}_{\tau}^{k+1} - \mathbf{f}_{\tau}^k\|_{L^2}^2 + \frac{\tau}{2} \|\mathbf{u}_{\tau}^k\|_{L^2}^2 \right) \\ & \quad + \frac{\delta}{2} \|\mathbf{u}_{\tau}^j\|_{L^2}^2 + \frac{1}{2\delta} \|\mathbf{f}_{\tau}^j\|_{L^2}^2 + \|\mathbf{f}_{\tau}^1\|_{L^2} \|\mathbf{u}_{\tau}^0\|_{L^2}, \end{aligned} \quad (53)$$

for all $\delta > 0$.

Now, from (53), by making use of Lemma 2 and by recalling that $\mathbf{u}_{\tau}^0 = \mathbf{u}^0$ we arrive at

$$\sum_{k=1}^j \int_{\Omega} \mathbf{f}_{\tau}^k \cdot (\mathbf{u}_{\tau}^k - \mathbf{u}_{\tau}^{k-1}) dx \leq \sum_{k=1}^{j-1} \frac{\tau}{2} \|\mathbf{u}_{\tau}^k\|_{L^2}^2 + \frac{\delta}{2} \|\mathbf{u}_{\tau}^j\|_{L^2}^2 + C_{\delta}. \quad (54)$$

On combining (52) with (54), and on using subsequently Korn's, Holder's, Young's, and Poincaré's inequalities, on choosing δ sufficiently small, we obtain

$$\|\mathbf{u}_{\tau}^j\|_{H^1}^2 + \tau \sum_{k=1}^j \|p_{\tau}^k\|_{H^1}^2 \leq C \left(1 + \tau \sum_{k=1}^{j-1} \|\mathbf{u}_{\tau}^k\|_{H^1}^2 \right). \quad (55)$$

We now use the discrete Gronwall inequality in Lemma 1 with $a_0 = C$ and $b_j = C\tau$ to obtain the following estimate:

$$\|\mathbf{u}_\tau^j\|_{H^1}^2 \leq Ce^{C\tau(j-1)} \leq Ce^{CT}, \quad (56)$$

whence (49a). By (56) the right-hand side of (55) is bounded uniformly with respect to τ , and we conclude that

$$\int_0^T \|\bar{p}_\tau(t)\|_{H^1}^2 dt = \tau \sum_{k=1}^n \|p_\tau^k\|_{H^1}^2 \leq C, \quad (57)$$

which entails (49b). \square

Proposition 3 (Time regularity). *The estimates*

$$\|\dot{\mathbf{u}}_\tau\|_{L^2(0,T;H^1)}^2 + \|\nabla \bar{p}_\tau\|_{L^\infty(0,T;L^2)}^2 \leq C, \quad (58a)$$

$$\|\dot{p}_\tau\|_{L^2(0,T;L^2)}^2 \leq C, \quad (58b)$$

hold uniformly with respect to τ .

Proof. Thanks to the definition of \mathbf{u}_τ^0 and p_τ^0 given in (36), we have that \mathbf{u}_τ^0 satisfies the weak form of the discrete force balance (37a) and the boundary condition (38a) for $k = 0$. Thus, for all $k = 1 \dots n$ we have:

$$\begin{cases} \int_\Omega \left(\mathbb{C} \frac{\mathbf{E}\mathbf{u}_\tau^k - \mathbf{E}\mathbf{u}_\tau^{k-1}}{\tau} \cdot \mathbf{E}\mathbf{v} - \frac{p_\tau^k - p_\tau^{k-1}}{\tau} \operatorname{div} \mathbf{v} \right) dx = \int_\Omega \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \cdot \mathbf{v} dx, \\ \int_\Omega (\operatorname{div} \mathbf{u}_\tau^k q + \tau \mathbf{K} \nabla p_\tau^k \cdot \nabla q) dx = \int_\Omega \operatorname{div} \mathbf{u}_\tau^{k-1} q dx, \\ \text{for all } \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3) \text{ and } q \in H_{p,D}^1(\Omega). \end{cases} \quad (59)$$

On choosing $\mathbf{v} = (\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1})/\tau$ and $q = (p_\tau^k - p_\tau^{k-1})/\tau^2$ as test functions in (59) (the latter choice is legal also for $k = 1$, since $p_0 \in H_{p,D}^1(\Omega)$) and on adding the resulting equations we obtain, after making use of some elementary inequalities,

$$\begin{aligned} c \left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{H^1}^2 + \frac{c}{\tau} \left(\|\nabla p_\tau^k\|_{L^2}^2 - \|\nabla p_\tau^{k-1}\|_{L^2}^2 \right) \\ \leq \frac{1}{2\delta} \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2}^2 + \frac{\delta}{2} \left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{L^2}^2, \end{aligned} \quad (60)$$

where $c > 0$ depends on the constant of Korn's inequality and on the elasticity tensor \mathbb{C} . On choosing $\delta = c$ we deduce,

$$\begin{aligned} \frac{c}{2} \sum_{k=1}^j \tau \left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{H^1}^2 + \|\nabla p_\tau^j\|_{L^2}^2 \\ \leq \|\nabla p_0\|_{L^2}^2 + \frac{1}{2c} \sum_{k=1}^j \tau \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2}^2, \end{aligned} \quad (61)$$

for every $j \in \{1, 2, \dots, N\}$. The bound (58a) is then obtained by invoking Lemma 2, on noting that

$$\int_0^T \|\dot{\mathbf{u}}_\tau(t)\|_{H^1}^2 dt = \sum_{k=1}^n \tau \left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{H^1}^2.$$

We now prove (58b). From the first equation of (59), namely

$$\int_{\Omega} \left(\mathbb{C} \frac{\mathbf{E}\mathbf{u}_\tau^k - \mathbf{E}\mathbf{u}_\tau^{k-1}}{\tau} \cdot \mathbf{E}\mathbf{v} - \frac{p_\tau^k - p_\tau^{k-1}}{\tau} \operatorname{div} \mathbf{v} \right) dx = \int_{\Omega} \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \cdot \mathbf{v} dx,$$

which holds for all $\mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3)$, we find

$$\begin{aligned} \int_{\Omega} \frac{p_\tau^k - p_\tau^{k-1}}{\tau} \operatorname{div} \mathbf{v} dx &\leq C \left\| \frac{\mathbf{E}\mathbf{u}_\tau^k - \mathbf{E}\mathbf{u}_\tau^{k-1}}{\tau} \right\|_{L^2} \|\mathbf{E}\mathbf{v}\|_{L^2} + \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2} \|\mathbf{v}\|_{L^2} \\ &\leq C \left(\left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{H^1} + \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2} \right) \|\mathbf{v}\|_{H^1}. \end{aligned} \quad (62)$$

Let $\varphi \in L^2(\Omega)$ with $\|\varphi\|_{L^2} \leq 1$. Then by [2, Lemma 3.2], there exists a $\mathbf{v}_\varphi \in H_{u,D}^1(\Omega; \mathbb{R}^3)$ such that $\operatorname{div} \mathbf{v}_\varphi = \varphi$ and $\|\mathbf{v}_\varphi\|_{H^1} \leq C\|\varphi\|_{L^2} \leq C$. By taking $\mathbf{v} = \mathbf{v}_\varphi$ in (62) we are led to

$$\int_{\Omega} \frac{p_\tau^k - p_\tau^{k-1}}{\tau} \varphi dx \leq C \left(\left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{H^1} + \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2} \right),$$

which holds for every $\varphi \in L^2(\Omega)$ with $\|\varphi\|_{L^2} \leq 1$. Thus

$$\left\| \frac{p_\tau^k - p_\tau^{k-1}}{\tau} \right\|_{L^2} \leq C \left(\left\| \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \right\|_{H^1} + \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2} \right).$$

Taking squares, multiplying by τ , and summing over k we find

$$\sum_{k=1}^N \tau \left\| \frac{p_\tau^k - p_\tau^{k-1}}{\tau} \right\|_{L^2}^2 \leq C \left(\|\dot{\mathbf{u}}_\tau\|_{L^2(0,T;H^1)}^2 + \sum_{k=1}^n \tau \left\| \frac{\mathbf{f}_\tau^k - \mathbf{f}_\tau^{k-1}}{\tau} \right\|_{L^2}^2 \right),$$

and taking into account Lemma 2 and (58a) we obtain (58b). \square

4.3 Passage to the limit.

Proposition 4 (Converging subsequences). *There exist*

$$\mathbf{u} \in H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3))$$

and

$$p \in L^\infty(0, T; H_{p,D}^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

with $p(0) = p_0$ and $\mathbf{u}(0) = \mathbf{u}_0$ where \mathbf{u}_0 is defined in (35), such that, up to subsequences,

$$\begin{aligned} \bar{\mathbf{u}}_\tau &\xrightarrow{*} \mathbf{u} \text{ in } L^\infty(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)) & \mathbf{u}_\tau &\rightharpoonup \mathbf{u} \text{ in } H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ \bar{p}_\tau &\rightharpoonup p \text{ in } L^2(0, T; H_{p,D}^1(\Omega)), & p_\tau &\rightharpoonup p \text{ in } H^1(0, T; L^2(\Omega)). \end{aligned}$$

Proof. We start by proving that

$$\bar{p}_\tau - p_\tau \rightarrow 0, \quad \text{in } L^2(0, T; L^2(\Omega)). \quad (63)$$

Indeed, using the definitions (48), we have that

$$\begin{aligned} \|\bar{p}_\tau - p_\tau\|_{L^2(0,T;L^2)}^2 &= \sum_{k=1}^n \int_{(k-1)\tau}^{k\tau} \|\bar{p}_\tau(t) - p_\tau(t)\|_{L^2}^2 dt \\ &= \sum_{k=1}^n \int_{(k-1)\tau}^{k\tau} \|p_\tau^k - p_\tau^{k-1}\|_{L^2}^2 \left(\frac{k\tau - t}{\tau}\right)^2 dt \\ &\leq \sum_{k=1}^n \tau \|p_\tau^k - p_\tau^{k-1}\|_{L^2}^2 \\ &= \tau^2 \|\dot{p}_\tau\|_{L^2(0,T;L^2)}^2 \leq C\tau^2, \end{aligned}$$

where the last estimate follows from (58b). Similarly we prove that

$$\bar{\mathbf{u}}_\tau - \mathbf{u}_\tau \rightarrow 0, \quad \text{in } L^2(0, T; H^1(\Omega)). \quad (64)$$

Thanks to Proposition 2 there exist two subsequences (which we do not relabel) such that

$$\begin{aligned} \bar{\mathbf{u}}_\tau &\xrightarrow{*} \mathbf{u} \text{ in } L^\infty(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ \bar{p}_\tau &\rightharpoonup p \text{ in } L^2(0, T; H_{p,D}^1(\Omega)). \end{aligned}$$

By Proposition 3 we also have that

$$\nabla \bar{p}_\tau \xrightarrow{*} \nabla p \text{ in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$$

and hence, by Poincaré inequality, $p \in L^\infty(0, T; H_{p,D}^1(\Omega))$. Also, by (63), (64), and Proposition 3 we deduce that

$$\begin{aligned} \mathbf{u}_\tau &\rightharpoonup \mathbf{u} \text{ in } H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ p_\tau &\rightharpoonup p \text{ in } H^1(0, T; L^2(\Omega)). \end{aligned}$$

Finally, since $p_\tau(0) = p_0$ and $\mathbf{u}_\tau(0) = \mathbf{u}_0$ we have, from the convergence above, that $p(0) = p_0$ and $\mathbf{u}(0) = \mathbf{u}_0$, as required. \square

4.4 Proof of Theorem 1

Let $\bar{\mathbf{f}}_\tau$ be the piecewise constant interpolation of \mathbf{f}_τ^k , as in (48). From assumption (32c) and definition (34) it follows that

$$\bar{\mathbf{f}}_\tau \rightarrow \mathbf{f} \quad \text{in } L^2(0, T; L^2(\Omega)).$$

Choose any test function $\varphi \in \mathcal{D}(0, T)$. From the first of (39) we obtain

$$\int_0^T \int_\Omega (\mathbb{C} \mathbf{E} \bar{\mathbf{u}}_\tau(t) \cdot \mathbf{E} \mathbf{v} - \bar{p}_\tau(t) \operatorname{div} \mathbf{v}) \, dx \, \varphi(t) \, dt = \int_0^T \int_\Omega \bar{\mathbf{f}}_\tau(t) \cdot \mathbf{v} \, dx \, \varphi(t) \, dt,$$

where \mathbf{v} is an arbitrary function in $H_{u,D}^1(\Omega; \mathbb{R}^3)$. Recalling Proposition 7 and on passing to the limit we obtain

$$\int_0^T \left(\int_\Omega (\mathbb{C} \mathbf{E} \mathbf{u}(t) \cdot \mathbf{E} \mathbf{v} - p(t) \operatorname{div} \mathbf{v}) \, dx \right) \varphi(t) \, dt = \int_0^T \left(\int_\Omega \mathbf{f}(t) \cdot \mathbf{v} \, dx \right) \varphi(t) \, dt,$$

whence, by the arbitrariness of φ we have, for a.e. $t \in (0, T)$,

$$\int_\Omega (\mathbb{C} \mathbf{E} \mathbf{u}(t) \cdot \mathbf{E} \mathbf{v} - p(t) \operatorname{div} \mathbf{v}) \, dx = \int_\Omega \mathbf{f}(t) \cdot \mathbf{v} \, dx. \quad (65)$$

Since $\mathbf{u}(0) = \mathbf{u}_0$ with \mathbf{u}_0 solving (35), the equation (65) holds also at $t = 0$. Moreover, the continuity of \mathbf{u} , p , and \mathbf{f} implies that (65) holds for all $t \in [0, T)$.

The second of (39) yields

$$\int_0^T \int_\Omega (\operatorname{div} \dot{\mathbf{u}}_\tau q + \mathbf{K} \nabla \bar{p}_\tau \cdot \nabla q) \, dx \, \varphi(t) \, dt = 0.$$

Again, recalling Proposition 7 and passing to the limit we obtain

$$\int_0^T \int_{\Omega} (\operatorname{div} \dot{\mathbf{u}} q + \mathbf{K} \nabla p \cdot \nabla q) \, dx \, \varphi(t) \, dt = 0,$$

and by localizing we obtain

$$\int_{\Omega} (\operatorname{div} \dot{\mathbf{u}} q + \mathbf{K} \nabla p \cdot \nabla q) \, dx = 0 \quad \text{for a.e. } t \in (0, T).$$

We next address the uniqueness of the solution. We choose $\mathbf{v} = \dot{\mathbf{u}}(t)$ and $q = p(t)$ in the weak formulation (77), we add the resulting equations and for $s \in (0, T]$ we integrate by parts over the time interval $(0, s)$ to obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \mathbb{C} \mathbf{E} \mathbf{u}(s) \cdot \mathbf{E} \mathbf{u}(s) \, dx + \int_0^s \int_{\Omega} \mathbf{K} \nabla p(t) \cdot \nabla p(t) \, dx \, dt \\ = \frac{1}{2} \int_{\Omega} \mathbb{C} \mathbf{E} \mathbf{u}(0) \cdot \mathbf{E} \mathbf{u}(0) \, dx + \int_0^s \int_{\Omega} \mathbf{f}(t) \cdot \dot{\mathbf{u}}(t) \, dx \, dt. \end{aligned} \quad (66)$$

A further integration by parts on the right-hand side of (66) yields

$$\mathcal{E}(s) + \int_0^s \int_{\Omega} \mathbf{K} \nabla p(t) \cdot \nabla p(t) \, dx \, dt = \mathcal{E}(0) - \int_0^s \int_{\Omega} \dot{\mathbf{f}}(t) \cdot \mathbf{u}(t) \, dx \, dt, \quad (67)$$

where

$$\mathcal{E}(s) := \int_{\Omega} \left(\frac{1}{2} \mathbb{C} \mathbf{E} \mathbf{u}(s) \cdot \mathbf{E} \mathbf{u}(s) - \mathbf{f}(s) \cdot \mathbf{u}(s) \right) \, dx, \quad (68)$$

The coercivity of the elasticity tensor \mathbb{C} assumed in (32f), Young's inequality, and Korn's inequality yield

$$\begin{aligned} \mathcal{E}(s) &\geq \frac{c_{\mathbb{C}}}{2} \|\mathbf{E} \mathbf{u}(s)\|_{L^2}^2 - \frac{1}{\delta} \|\mathbf{f}(s)\|_{L^2}^2 - \delta \|\mathbf{u}(s)\|_{L^2}^2 \\ &\geq c_1 \|\mathbf{u}(s)\|_{H^1}^2 - \frac{C_1}{\delta} \left(\|\mathbf{f}(0)\|_{L^2}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2)}^2 \right) - \delta \|\mathbf{u}(s)\|_{H^1}^2, \end{aligned}$$

where $c_1 := c_K c_{\mathbb{C}}/2$, with $c_K := \inf \{ \|\mathbf{E} \mathbf{u}\|_{L^2} / \|\mathbf{u}\|_{H^1}, \mathbf{u} \neq \mathbf{0} \} > 0$ and $C_1 = 2(1+T)$. For $\delta = c_1/2$ the above chain of inequalities yields

$$\|\mathbf{u}(s)\|_{H^1}^2 \leq C_2 \left(\mathcal{E}(s) + \|\mathbf{f}(0)\|_{L^2}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2)}^2 \right), \quad (69)$$

where $C_2 = \max \left(\frac{2}{c_1}, 4 \frac{C_1}{c_1^2} \right)$. Moreover, it follows from Poincaré's inequality that

$$\int_0^s \|p(t)\|_{H^1}^2 \, dt \leq C_3 \int_0^s \int_{\Omega} \mathbf{K} \nabla p(t) \cdot \nabla p(t) \, dx \, dt, \quad (70)$$

for a sufficiently large constant C_3 . It follows from (67), (69), and (70) that, for $C_4 = \max(C_2, C_3)$,

$$\begin{aligned} \|\mathbf{u}(s)\|_{H^1}^2 + \int_0^s \|p(t)\|_{H^1}^2 dt &\leq C_4 \left(\mathcal{E}(0) - \int_0^s \dot{\mathbf{f}}(t) \cdot \mathbf{u}(t) dt \right) + C_2 \left(\|\mathbf{f}(0)\|_{L^2}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2)}^2 \right) \\ &\leq C_5 \left(\mathcal{E}(0) + \int_0^s \|\mathbf{u}(t)\|_{H^1}^2 dt + \|\mathbf{f}(0)\|_{L^2}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2)}^2 \right), \end{aligned}$$

where C_5 is a suitably large constant. Consider now two solutions, say (\mathbf{u}_1, p_1) and (\mathbf{u}_2, p_2) , and let $\delta\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $\delta p = p_1 - p_2$. The pair $(\delta\mathbf{u}, \delta p)$ is a solution of (77) with homogeneous forcing $\mathbf{f} = \mathbf{0}$, and with homogeneous initial condition $\delta p(0) = 0$. The uniqueness of the solution of the elasticity problem implies that $\delta\mathbf{u}(0) = \mathbf{0}$. Thus, for such solution, the estimate (71) holds with $\mathcal{E}(0) = 0$ and $\mathbf{f} = \mathbf{0}$, namely,

$$\|\delta\mathbf{u}(s)\|_{H^1}^2 + \int_0^s \|\delta p(t)\|_{H^1}^2 dt \leq \frac{C_4}{2} \int_0^s \|\delta\mathbf{u}(t)\|_{H^1}^2 dt. \quad (71)$$

From (71) and Gronwall's inequality we obtain that $\delta\mathbf{u} = 0$ and $\delta p = 0$, thus $\mathbf{u}_1 = \mathbf{u}_2$ and $p_1 = p_2$, as required.

Remark. Taking into account the applied pressure p_a . If the simplifying assumption (31) is removed, the existence and uniqueness of a weak solution to (28)–(30) can be proved with little conceptual difficulty, provided that the time-dependent pressure field $p_a(t)$ prescribed on $\Gamma_{p,D}$ satisfies

$$p_a \in H^1(0, T; H^{1/2}(\Gamma_{p,D})). \quad (72)$$

Indeed, let $\tilde{p}_a(t)$ be the lifting of $p_a(t)$ to Ω defined by

$$\begin{aligned} \operatorname{div} \mathbf{K} \nabla \tilde{p}_a(t) &= 0 && \text{in } \Omega, \\ \tilde{p}_a(t) &= p_a(t) && \text{on } \Gamma_{p,D}, \\ -\mathbf{K} \nabla \tilde{p}_a(t) \cdot \mathbf{n} &= 0 && \text{on } \Gamma_{p,N}. \end{aligned} \quad (73)$$

Problem (28)–(30) can be reformulated in terms of the unknowns $\mathbf{u}(t)$ and

$$\hat{p}(t) := p(t) - \tilde{p}_a(t)$$

to obtain the following system:

$$\operatorname{div} \mathbb{C} \mathbf{E} \mathbf{u} - \nabla \hat{p} = -\mathbf{f} + \nabla \tilde{p}_a, \quad (74a)$$

$$\operatorname{div} \dot{\mathbf{u}} - \operatorname{div} \mathbf{K} \nabla \hat{p} = 0, \quad (74b)$$

with boundary conditions:

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_{u,D}, \quad (\mathbb{C}\mathbf{E}\mathbf{u}(t))\mathbf{n} - \widehat{p}(t)\mathbf{n} = \tilde{p}_a(t)\mathbf{n} \text{ on } \Gamma_{u,N}, \quad (75a)$$

$$\widehat{p}(t) = 0 \text{ on } \Gamma_{p,D}, \quad -\mathbf{K}\nabla\widehat{p}(t) \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{p,N}, \quad (75b)$$

and with the initial condition

$$\widehat{p}(0) = \widehat{p}_0 := p_0 - \tilde{p}_a. \quad (76)$$

The weak formulation of (74)–(76) is

$$\left\{ \begin{array}{l} \mathbf{u} \in H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ \widehat{p} \in L^\infty(0, T; H_{p,D}^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \text{with} \quad \widehat{p}(0) = \widehat{p}_0, \\ \int_{\Omega} (\mathbb{C}\mathbf{E}\mathbf{u}(t) \cdot \mathbf{E}\mathbf{v} - p(t) \operatorname{div} \mathbf{v}) \, dx = \langle \boldsymbol{\ell}(t), \mathbf{v} \rangle, \\ \quad \forall \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3), \forall t \in (0, T), \\ \int_{\Omega} (\dot{\mathbf{u}}(t)q + \mathbf{K}\nabla p(t) \cdot \nabla q) \, dx = 0 \quad \forall q \in H_{p,D}^1(\Omega), \text{ for a.e. } t \in (0, T). \end{array} \right. \quad (77)$$

where the time-dependent linear functional $\boldsymbol{\ell} \in H^1(0, T; (H_{u,D}^1(\Omega; \mathbb{R}^3))')$ is defined by

$$\langle \boldsymbol{\ell}(t), \mathbf{v} \rangle := \int_{\Omega} (\mathbf{f}(t) \cdot \mathbf{v} + \tilde{p}_a \operatorname{div} \mathbf{v}) \, dx.$$

The proof of existence and uniqueness requires only minor conceptual changes. Namely, every occurrence of a scalar product $(\mathbf{f}(t), \mathbf{v}) = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} \, dx$ must be replaced with the pairing $\langle \boldsymbol{\ell}(t), \mathbf{v} \rangle$, and every occurrence of the Cauchy–Schwartz inequality $(\mathbf{f}(t), \mathbf{v}) \leq \|\mathbf{f}(t)\|_{L^2} \|\mathbf{v}\|_{L^2}$ must be replaced with $\langle \boldsymbol{\ell}(t), \mathbf{v} \rangle \leq \|\mathbf{f}(t)\|_{(H_{u,D}^1)'} \|\mathbf{v}\|_{H_{u,D}^1}$.

5 Plate equations

A family of shrinking plate-like domains. For $\varepsilon > 0$ a parameter that tends to null, we consider a slab of thickness εh modeled on a plane domain ω :

$$\Omega_\varepsilon = \omega \times \left(-\varepsilon \frac{h}{2}, +\varepsilon \frac{h}{2} \right), \quad \omega \subset \mathbb{R}^2. \quad (78)$$

We fix a part $\gamma_{u,D} \subset \partial\omega$, we prescribe null displacement on

$$\Gamma_{u,D,\varepsilon} = \gamma_{u,D} \times \left(-\varepsilon \frac{h}{2}, +\varepsilon \frac{h}{2} \right), \quad (79)$$

and null traction on

$$\Gamma_{u,N,\varepsilon} = \partial\Omega_\varepsilon \setminus \Gamma_{u,D,\varepsilon}.$$

On the top and bottom faces of the slab

$$\Gamma_{p,D,\varepsilon} = \omega \times \left\{ \pm \varepsilon \frac{h}{2} \right\} \quad (80)$$

we impose a pressure field $p_{a,\varepsilon}$. On the lateral side of the plate

$$\Gamma_{p,N,\varepsilon} = \partial\omega \times \left(-\varepsilon \frac{h}{2}, +\varepsilon \frac{h}{2} \right) \quad (81)$$

we require that the flux be null.

A family of evolution problems. For each ε we consider the following system

$$\operatorname{div} \mathbb{C}_\varepsilon \mathbf{E} \mathbf{u}_\varepsilon - \nabla p_\varepsilon = -\mathbf{f}_\varepsilon, \quad (82a)$$

$$\operatorname{div} \dot{\mathbf{u}}_\varepsilon - \operatorname{div} \mathbf{K}_\varepsilon \nabla p_\varepsilon = 0, \quad (82b)$$

in the unknowns \mathbf{u}_ε and p_ε , formulated in the space-time domain $\Omega_\varepsilon \times (0, T)$, with boundary conditions

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_{u,D,\varepsilon}, \quad (\mathbb{C}_\varepsilon \mathbf{E} \mathbf{u}_\varepsilon(t)) \mathbf{n} - p_\varepsilon(t) \mathbf{n} = 0 \text{ on } \Gamma_{u,N,\varepsilon}, \quad (83a)$$

$$p_\varepsilon(t) = p_{a,\varepsilon}(t) \text{ on } \Gamma_{p,D,\varepsilon}, \quad -\mathbf{K}_\varepsilon \nabla p_\varepsilon(t) \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{p,N,\varepsilon}, \quad (83b)$$

and with the initial condition

$$p_\varepsilon(0) = p_{0,\varepsilon} \quad (84)$$

We make the following assumptions

$$\mathcal{H}^1(\gamma_{u,D}) > 0 \quad (85a)$$

$$p_{0,\varepsilon} \in H_{p,D,\varepsilon}^1(\Omega_\varepsilon), \quad p_{0,\varepsilon} = p_{a,\varepsilon} \text{ on } \Gamma_{p,D,\varepsilon}, \quad (85b)$$

$$\mathbf{f}_\varepsilon \in H^1(0, T; L^2(\Omega_\varepsilon; \mathbb{R}^3)), \quad (85c)$$

$$(\mathbb{C}_\varepsilon)_{ijkl} \in L^\infty(\Omega_\varepsilon), \quad (85d)$$

$$(\mathbb{C}_\varepsilon)_{ijkl} = (\mathbb{C}_\varepsilon)_{klij} = (\mathbb{C}_\varepsilon)_{jikl}, \quad (\mathbb{C}_\varepsilon)_{333\alpha} = (\mathbb{C}_\varepsilon)_{3\alpha\beta\gamma} = 0, \quad (85e)$$

$$p_{a,\varepsilon} \in H^1(0, T; H^{1/2}(\Gamma_{p,D,\varepsilon})), \quad (85f)$$

$$\exists c_{\mathbb{C}} > 0 : \mathbb{C}_\varepsilon \mathbf{A} \cdot \mathbf{A} \geq c_{\mathbb{C}} |\mathbf{A}|^2 \quad \forall \varepsilon > 0, \mathbf{A} \in \mathbb{R}_{\text{Sym}}^{3 \times 3} \quad \text{a.e. in } \Omega_\varepsilon, \quad (85g)$$

$$\mathbf{K}_\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}_{\text{Sym}}^{3 \times 3}), \quad (K_\varepsilon)_{3\alpha} = 0. \quad (85h)$$

$$\exists c_{\mathbf{K}} > 0 : \mathbf{K}_\varepsilon \mathbf{a} \cdot \mathbf{a} \geq c_{\mathbf{K}} |\mathbf{a}|^2 \quad \forall \varepsilon, \forall \mathbf{a} \in \mathbb{R}^3 \quad \text{a.e. in } \Omega_\varepsilon. \quad (85i)$$

As pointed out in the remark at the conclusion of the previous section, it is convenient to decompose the pressure p_ε into the sum of

- a fluctuating component \widehat{p}_ε , that is coupled with the displacement \mathbf{u} and
- a component $\tilde{p}_{a,\varepsilon}$ that is directly controlled through the applied pressure $p_{a,\varepsilon}$.

To this effect, we set

$$p_\varepsilon(t) = \widehat{p}_\varepsilon(t) + \tilde{p}_{a,\varepsilon}(t),$$

where $\tilde{p}_{a,\varepsilon}$ is the lifting of the boundary datum to Ω_ε obtained by solving the following (time-dependent) boundary value problem:

$$\begin{aligned} \operatorname{div} \mathbf{K}_\varepsilon \nabla \tilde{p}_{a,\varepsilon}(t) &= 0 & \text{in } \Omega_\varepsilon, \\ \tilde{p}_{a,\varepsilon}(t) &= p_{a,\varepsilon}(t) & \text{on } \Gamma_{p,D,\varepsilon}, \\ -\mathbf{K}_\varepsilon \nabla \tilde{p}_{a,\varepsilon}(t) \cdot \mathbf{n} &= 0 & \text{on } \Gamma_{p,N,\varepsilon}. \end{aligned} \quad (86)$$

We introduce the notation $H_{p,D,\varepsilon}^1(\Omega_\varepsilon) = \{q \in H^1(\Omega_\varepsilon) : q = 0 \text{ on } \Gamma_{p,D,\varepsilon}\}$. At a given $t \in [0, T]$, the weak form of (86) is

$$\left\{ \begin{array}{l} p_{\varepsilon,a}(t) \in H^1(\Omega_\varepsilon) \quad \text{with} \quad \tilde{p}_a^\varepsilon(t) = p_a^\varepsilon(t) \text{ on } \Gamma_{p,D,\varepsilon}, \\ \int_{\Omega_\varepsilon} \mathbf{K}_\varepsilon \nabla \tilde{p}_{a,\varepsilon}(t) \cdot \nabla q \, dx = 0 \quad \forall q \in H_{p,D,\varepsilon}^1(\Omega_\varepsilon). \end{array} \right. \quad (87)$$

The weak formulation of the problem governing $(\mathbf{u}_\varepsilon, \widehat{p}_\varepsilon)$ is the following:

$$\left\{ \begin{array}{l} \mathbf{u}_\varepsilon \in H^1(0, T; H_{u,D,\varepsilon}^1(\Omega_\varepsilon; \mathbb{R}^3)), \\ \widehat{p}_\varepsilon \in L^\infty(0, T; H_{p,D,\varepsilon}^1(\Omega_\varepsilon)) \cap H^1(0, T; L^2(\Omega_\varepsilon)) \quad \text{with} \quad \widehat{p}_\varepsilon(0) = \widehat{p}_{0,\varepsilon},^1 \\ \int_{\Omega_\varepsilon} (\mathbb{C}_\varepsilon \mathbf{E} \mathbf{u}_\varepsilon(t) \cdot \mathbf{E} \mathbf{v} - \widehat{p}_\varepsilon(t) \operatorname{div} \mathbf{v}) \, dx = \int_{\Omega_\varepsilon} (\mathbf{f}_\varepsilon(t) \cdot \mathbf{v} + \tilde{p}_{a,\varepsilon} \operatorname{div} \mathbf{v}) \, dx \\ \quad \forall \mathbf{v} \in H_{u,D,\varepsilon}^1(\Omega_\varepsilon; \mathbb{R}^3), \forall t \in [0, T], \\ \int_{\Omega_\varepsilon} \operatorname{div} \dot{\mathbf{u}}_\varepsilon(t) q + \mathbf{K}_\varepsilon \nabla \widehat{p}_\varepsilon(t) \cdot \nabla q \, dx = 0 \quad \forall q \in H^1(\Omega_\varepsilon), \text{ for a.e. } t \in (0, T). \end{array} \right. \quad (88)$$

Here we use the notation $H_{u,D,\varepsilon}^1(\Omega_\varepsilon; \mathbb{R}^3) = \{\mathbf{v} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{u,D,\varepsilon}\}$. Moreover,

$$\widehat{p}_{0,\varepsilon} = p_{0,\varepsilon} - \tilde{p}_{a,\varepsilon}(0).$$

The existence and uniqueness of a weak solution follows from Theorem 1 and from the remarks at the end of the previous section.

Change of independent variables. We next reformulate the problem on a domain that does not depend on ε . An obvious choice is

$$\Omega = \omega \times \left(-\frac{h}{2}, +\frac{h}{2}\right),$$

which corresponds to taking $\varepsilon = 1$ in (78). To this effect, we introduce the linear map

$$\mathbf{r}_\varepsilon : \Omega_\varepsilon \rightarrow \Omega, \quad \mathbf{r}_\varepsilon(\mathbf{x}) := \mathbf{R}_\varepsilon \mathbf{x}, \quad \mathbf{R}_\varepsilon := \text{diag}(1, 1, \varepsilon^{-1})$$

(i.e. \mathbf{R}^ε is the diagonal matrix with entries 1, 1, and ε^{-1}). We can therefore write the typical point \mathbf{x} of the fixed domain Ω as

$$\mathbf{x} = \mathbf{R}_\varepsilon \mathbf{x}_\varepsilon$$

where \mathbf{x}_ε is a typical point of the shrinking domain Ω_ε .

Change of unknowns. At this stage, we express the fields \mathbf{u}_ε and p_ε in terms of \mathbf{x} and write the corresponding evolution problem on the space-time domain $\Omega \times (0, T)$. The result will be a singular-perturbation problem to study through asymptotic analysis. Before doing that, however, we go through an additional step: we change the dependent variables that so that it is easier to extract information from the resulting singular perturbation problem. Our choice of the new dependent variables is suggested by known results concerning the purely mechanical problem of bending of a linearly elastic plate. For this problem it is known [11] that in the limit as ε tends to null the ratio between in-plane displacement and transverse displacement is of the order of ε . This motivates the introduction of the scaled displacement defined by:

$$u_\alpha^\varepsilon(\mathbf{x}, t) := \frac{1}{\varepsilon} (\mathbf{u}_\varepsilon)_\alpha(\mathbf{x}^\varepsilon, t) \quad u_3^\varepsilon(\mathbf{x}, t) := (\mathbf{u}_\varepsilon)_3(\mathbf{x}^\varepsilon, t) \quad (89)$$

(henceforth Greek free indices range between 1 and 2). As to the pressure, we select the following change of variable:

$$\tilde{p}^\varepsilon(\mathbf{x}, t) = \frac{1}{\varepsilon} \hat{p}_\varepsilon(\mathbf{x}^\varepsilon, t), \quad \tilde{p}_a^\varepsilon(\mathbf{x}, t) = \frac{1}{\varepsilon} \tilde{p}_{a,\varepsilon}(\mathbf{x}^\varepsilon, t). \quad (90)$$

One checks that $\mathbf{u}^\varepsilon(t) = \varepsilon \mathbf{R}_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x}, t) \circ \mathbf{r}^\varepsilon$. This yields $\nabla \mathbf{u}^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon \mathbf{R}_\varepsilon \nabla \mathbf{u}_\varepsilon(\mathbf{x}, t) \mathbf{R}_\varepsilon = \nabla^\varepsilon \mathbf{u}^\varepsilon$, that is

$$\nabla \mathbf{u}_\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon \nabla^\varepsilon \mathbf{u}^\varepsilon(\mathbf{x}, t) \quad \text{where} \quad \nabla^\varepsilon \mathbf{u}^\varepsilon := \mathbf{R}_\varepsilon \nabla \mathbf{u}^\varepsilon \mathbf{R}_\varepsilon = \begin{pmatrix} \partial_{\alpha\beta} \mathbf{u}^\varepsilon & \frac{\partial_\alpha u_3^\varepsilon}{\varepsilon} \\ \frac{\partial_\beta u_3^\varepsilon}{\varepsilon} & \frac{\partial_3 u_3^\varepsilon}{\varepsilon^2} \end{pmatrix}, \quad (91)$$

whence, in particular,

$$\mathbf{E}\mathbf{u}^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon \mathbf{E}^\varepsilon \mathbf{u}_\varepsilon(\mathbf{x}, t), \quad \text{where} \quad \mathbf{E}^\varepsilon \mathbf{u}_\varepsilon := \text{sym } \nabla^\varepsilon \mathbf{u}_\varepsilon, \quad (92)$$

and

$$\text{div } \mathbf{u}_\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon \text{div}^\varepsilon \mathbf{u}^\varepsilon(\mathbf{x}, t) \quad \text{where} \quad \text{div}^\varepsilon \mathbf{u}^\varepsilon := \text{tr } \nabla^\varepsilon \mathbf{u}^\varepsilon. \quad (93)$$

Similarly, the gradient of the fluctuating part \widehat{p}_ε of the pressure can be rendered in terms of the spatial derivatives of new unknown \widehat{p}^ε through

$$\nabla \widehat{p}_\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon \nabla^\varepsilon \widehat{p}^\varepsilon(\mathbf{x}, t) \quad \text{where} \quad \nabla^\varepsilon \widehat{p}^\varepsilon := \begin{pmatrix} \partial_\alpha \widehat{p}^\varepsilon \\ \frac{\partial_3 \widehat{p}^\varepsilon}{\varepsilon} \end{pmatrix}, \quad (94)$$

with a similar formula holding for $\nabla \tilde{p}_\varepsilon$.

The singular perturbation problem. We introduce the shorthand notation $\Gamma_{u,D} \equiv \Gamma_{u,D,1}$, $\Gamma_{u,N} \equiv \Gamma_{u,N,1}$, $\Gamma_{p,D} \equiv \Gamma_{p,D,1}$, $\Gamma_{p,N} \equiv \Gamma_{p,N,1}$ and we define

$$\begin{aligned} \mathbb{C}^\varepsilon(\mathbf{x}) &:= \mathbb{C}_\varepsilon(\mathbf{x}^\varepsilon), & \mathbf{K}^\varepsilon(\mathbf{x}) &:= \mathbf{K}_\varepsilon(\mathbf{x}^\varepsilon), \\ f_\alpha^\varepsilon(\mathbf{x}, t) &:= \varepsilon (f_\varepsilon)_\alpha(\mathbf{x}^\varepsilon, t), & f_3^\varepsilon(\mathbf{x}, t) &:= \varepsilon^2 (f_\varepsilon)_\alpha(\mathbf{x}^\varepsilon, t). \\ p_a^\varepsilon(\mathbf{x}, t) &:= p_{a,\varepsilon}(\mathbf{x}^\varepsilon, t). \end{aligned}$$

It is now easy to check that $\tilde{p}_{a,\varepsilon}(t)$ solves (90) if and only if $\tilde{p}_a^\varepsilon(t)$ solves:

$$\begin{cases} \tilde{p}_a^\varepsilon \in H^1(\Omega), & \tilde{p}_a^\varepsilon(t) = p_a^\varepsilon(t) \text{ on } \Gamma_{p,D}, \\ \int_\Omega \mathbf{K}^\varepsilon \nabla^\varepsilon \tilde{p}_a^\varepsilon(t) \cdot \nabla^\varepsilon q \, dx = 0 \quad \forall q \in H_{p,D}^1(\Omega). \end{cases} \quad (95)$$

Moreover, $(\mathbf{u}_\varepsilon, \widehat{p}_\varepsilon)$ solves Problem (88) if and only if $(\mathbf{u}^\varepsilon, \widehat{p}^\varepsilon)$ solves the following problem:

$$\begin{cases} \mathbf{u}^\varepsilon \in H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ \widehat{p}^\varepsilon \in L^\infty(0, T; H_{p,D}^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \text{with} \quad \widehat{p}^\varepsilon(0) = \widehat{p}_0,^2 \\ \int_\Omega (\mathbb{C}^\varepsilon \mathbf{E}^\varepsilon \mathbf{u}^\varepsilon(t) \cdot \mathbf{E}^\varepsilon \mathbf{v} - \widehat{p}^\varepsilon(t) \text{div}^\varepsilon \mathbf{v}) \, dx = \int_\Omega (\mathbf{f}^\varepsilon(t) \cdot \mathbf{v} + \tilde{p}_a^\varepsilon \text{div}^\varepsilon \mathbf{v}) \, dx \\ \quad \forall \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3), \forall t \in [0, T), \\ \int_\Omega \text{div}^\varepsilon \dot{\mathbf{u}}^\varepsilon(t) q + \mathbf{K}^\varepsilon \nabla^\varepsilon \widehat{p}^\varepsilon(t) \cdot \nabla^\varepsilon q \, dx = 0 \quad \forall q \in H_{p,D}^1(\Omega), \text{ for a.e. } t \in (0, T). \end{cases} \quad (96)$$

Further assumptions. We consider a regime when \mathbf{K}^ε tends to a limit \mathbf{K} as $\varepsilon \rightarrow 0$ in the following sense:

$$\mathbf{K}^\varepsilon \rightarrow \mathbf{K} \quad \text{in } L^\infty(\Omega; \mathbb{R}^{3 \times 3}). \quad (97)$$

We shall make the following assumptions concerning the other data:

$$\mathbb{C}^\varepsilon \rightarrow \mathbb{C} \quad \text{in } L^\infty(\Omega; \mathbb{R}^{3 \times 3 \times 3 \times 3}), \quad (98a)$$

$$\mathbf{f}^\varepsilon \rightarrow \mathbf{f} \quad \text{in } H^1(0, T; L^2(\Omega)), \quad (98b)$$

$$p_a^\varepsilon \rightarrow p_a \quad \text{in } H^1(0, T; H^{1/2}(\Gamma_{p,D})). \quad (98c)$$

Convergence of the pressure field. Our first step is to study the asymptotic behavior of the lifting \tilde{p}_a^ε . We show that in the limit \tilde{p}_a^ε approximates the linear interpolation \hat{p}_a^ε of the pressures applied at the top and at the bottom of the plate. In what follows we set:

$$p_a^{\varepsilon, \pm}(x_1, x_2, t) := p_a^\varepsilon(x_1, x_2, \pm h/2, t).$$

Also, we use the same notation for the limit p_a :

$$p_a^\pm(x_1, x_2, t) := p_a(x_1, x_2, \pm h/2, t).$$

We next define:

$$\zeta(x) := h \left(\frac{\mathcal{K}_{33}^{-1}(x)}{\mathcal{K}_{33}^{-1}(x_1, x_2, h/2)} - \frac{1}{2} \right) \quad \text{where} \quad \mathcal{K}_{33}^{-1}(x) := \int_{-h/2}^{x_3} \frac{1}{K_{33}(x_1, x_2, z)} dz.$$

Lemma 3. *Under assumptions (97), let $\tilde{p}_a^\varepsilon(t) \in H^1(\Omega)$ solve (95) for all $t \in [0, T]$. Then*

$$\tilde{p}_a^\varepsilon \rightharpoonup \tilde{p}_a \quad \text{in } H^1(0, T; L^2(\Omega)), \quad \partial_3 \tilde{p}_a^\varepsilon \rightharpoonup \partial_3 \tilde{p}_a \quad \text{in } H^1(0, T; L^2(\Omega)),$$

where

$$\tilde{p}_a = \frac{p^+(x_1, x_2) + p^-(x_1, x_2)}{2} + \zeta(x) \frac{p^+(x_1, x_2) - p^-(x_1, x_2)}{h}. \quad (99)$$

Remark 1. *If K_{33} does not depend on x_3 then $\zeta(x) = x_3$ and the representation formula (99) results in \tilde{p}_a being the affine interpolation between the pressures applied at the top and at the bottom of the plate.*

Proof of Lemma 3. By standard results on liftings of traces, there exists $\tilde{p}_a^\varepsilon \in H^1(0, T; H^1(\Omega))$ such that $\tilde{p}_a^\varepsilon(t) = p_a^\varepsilon(t)$ on $\Gamma_{p,D}$ for all $t \in [0, T]$ and

$$\|\tilde{p}_a^\varepsilon\|_{H^1(0,T;H^1)} \leq C \|p_a^\varepsilon\|_{H^1(0,T;H^{1/2}(\Gamma_{p,D}))}, \quad (100)$$

with C independent on ε . We take $q = \tilde{p}_a^\varepsilon - \check{p}_a^\varepsilon$ as test function in (95) and we integrate on $(0, T)$ to obtain $\int_0^T \int_\Omega |\nabla^\varepsilon \tilde{p}_a^\varepsilon|^2 dx dt = \int_0^T \int_\Omega \nabla^\varepsilon \tilde{p}_a^\varepsilon \cdot \nabla^\varepsilon \check{p}_a^\varepsilon dx dt$, whence the estimate

$$\|\nabla^\varepsilon \tilde{p}_a^\varepsilon\|_{L^2(0,T;L^2)} \leq \|\nabla^\varepsilon \check{p}_a^\varepsilon\|_{L^2(0,T;L^2)}. \quad (101)$$

Next, we differentiate (95) with respect to time and we take $q = \partial_t(\tilde{p}_a^\varepsilon - \check{p}_a^\varepsilon)$ as test function. Then we integrate with respect to time to get $\int_0^T \int_\Omega |\nabla^\varepsilon \partial_t \tilde{p}_a^\varepsilon|^2 dx dt = \int_0^T \int_\Omega \nabla^\varepsilon \partial_t \tilde{p}_a^\varepsilon \cdot \nabla^\varepsilon \partial_t \check{p}_a^\varepsilon dx dt$, which yields

$$\|\nabla^\varepsilon \partial_t \tilde{p}_a^\varepsilon\|_{L^2(0,T;L^2)} \leq \|\nabla^\varepsilon \partial_t \check{p}_a^\varepsilon\|_{L^2(0,T;L^2)}. \quad (102)$$

Putting together (101) and (102) we have

$$\|\nabla^\varepsilon \tilde{p}_a^\varepsilon\|_{H^1(0,T;L^2)} \leq \|\nabla^\varepsilon \check{p}_a^\varepsilon\|_{H^1(0,T;L^2)}, \quad (103)$$

an estimate that is best written by decomposing the rescaled pressure gradient $\nabla^\varepsilon \tilde{p}_a^\varepsilon$ into its transverse component $\varepsilon^{-1} \partial_3 \tilde{p}_a^\varepsilon$ and its plane component $\bar{\nabla} \tilde{p}_a^\varepsilon = (\partial_1 \tilde{p}_a^\varepsilon, \partial_2 \tilde{p}_a^\varepsilon)$:

$$\|\partial_3 \tilde{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2 + \|\varepsilon \bar{\nabla} \tilde{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2 \leq C \|\partial_3 \check{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2 + \|\varepsilon \bar{\nabla} \check{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2. \quad (104)$$

From (100) and (104) and from Assumption (98c) we obtain

$$\|\partial_3 \tilde{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2 + \|\varepsilon \bar{\nabla} \tilde{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2 \leq C. \quad (105)$$

From (98c), (105), and from Poincaré inequality we have

$$\|\tilde{p}_a^\varepsilon\|_{H^1(0,T;L^2)}^2 \leq C. \quad (106)$$

From the bounds (105) and (106) we conclude that there exists $\tilde{p}_a \in H^1(0, T; L^2(\Omega))$ such that $\partial_3 \tilde{p}_a \in H^1(0, T; L^2(\Omega))$ and

$$\tilde{p}_a^\varepsilon \rightharpoonup \tilde{p}_a \quad \text{and} \quad \partial_3 \tilde{p}_a^\varepsilon \rightharpoonup \partial_3 \tilde{p}_a \quad \text{in } H^1(0, T; L^2(\Omega)), \quad (107)$$

for some subsequence. Moreover, by the continuity of the trace operator,

$$\tilde{p}_a(x_1, x_2, \pm h/2, t) = p_a^\pm(x_1, x_2, t) \quad \forall t \in [0, T]. \quad (108)$$

Next, from (85h) and (95) we have, for all $t \in [0, T)$,

$$\int_{\Omega} K_{33}^{\varepsilon} \partial_3 \tilde{p}_a^{\varepsilon}(t) \partial_3 q \, dx + \varepsilon^2 \int_{\Omega} K_{\alpha\beta}^{\varepsilon} \tilde{p}_{a,\alpha}^{\varepsilon}(t) \cdot q_{,\beta} \, dx = 0 \quad \forall q \in H_{p,D}^1(\Omega). \quad (109)$$

Thanks to (105), (107), and to assumption (97), we can pass to the limit in (109) to obtain, for all $t \in [0, T)$,

$$\int_{\Omega} K_{33} \partial_3 \tilde{p}_a(t) \partial_3 q \, dx = 0, \quad \forall q \in H_{p,D}^1(\Omega),$$

which implies, because of (108), that the representation (99) holds. We conclude the proof by noting that since the limit \tilde{p}_a is uniquely determined the whole sequence converges. \square

We introduce the following space:

$$H_{KL}^1(\Omega; \mathbb{R}^3) := \{\mathbf{u} \in H_{u,D}^1(\Omega; \mathbb{R}^3) : E_{3i} \mathbf{u} = 0 \text{ a.e. in } \Omega\}.$$

Proposition 5 (Compactness). *There exist $\mathbf{u} \in H^1(0, T; H^1(\Omega; \mathbb{R}^3))$ and $\eta \in H^1(0, T; L^2(\Omega))$ such that*

$$\mathbf{u}^{\varepsilon} \rightharpoonup \mathbf{u} \quad \text{in } H^1(0, T; H^1(\Omega; \mathbb{R}^3)), \quad (110a)$$

$$E_{\alpha\beta}^{\varepsilon} \mathbf{u}^{\varepsilon} = E_{\alpha\beta} \mathbf{u}^{\varepsilon} \rightharpoonup E_{\alpha\beta} \mathbf{u} \quad \text{in } H^1(0, T; L^2(\Omega)), \quad (110b)$$

$$E_{33}^{\varepsilon} \mathbf{u}^{\varepsilon} = \frac{\partial_3 u_3^{\varepsilon}}{\varepsilon^2} \rightharpoonup \eta \quad \text{in } H^1(0, T; L^2(\Omega)), \quad (110c)$$

$$\widehat{p}^{\varepsilon} \xrightarrow{*} 0 \quad \text{in } L^{\infty}(0, T; H^1(\Omega)), \quad (110d)$$

for some subsequence. Moreover, the limit \mathbf{u} satisfies

$$\mathbf{u}(t) \in H_{KL}^1(\Omega; \mathbb{R}^3) \quad (111a)$$

for a.e. $t \in (0, T)$.

Proof. Differentiate the first equation in (96) and take $\mathbf{v} = \partial_t \mathbf{u}^{\varepsilon}$ to get, for a.e. $t \in (0, T)$,

$$\begin{aligned} & \int_{\Omega} (\mathbb{C}^{\varepsilon} \partial_t \mathbf{E}^{\varepsilon} \mathbf{u}^{\varepsilon}(t) \cdot \partial_t \mathbf{E}^{\varepsilon} \mathbf{u}^{\varepsilon}(t) - \partial_t \widehat{p}^{\varepsilon}(t) \partial_t \operatorname{div}^{\varepsilon} \mathbf{u}^{\varepsilon}(t) \, dx \\ &= \int_{\Omega} (\partial_t \widehat{\mathbf{f}}^{\varepsilon}(t) \cdot \partial_t \mathbf{u}^{\varepsilon}(t) + \partial_t \tilde{p}_{a,\varepsilon}(t) \partial_t \operatorname{div}^{\varepsilon} \mathbf{u}^{\varepsilon}(t)) \, dx. \end{aligned}$$

Choose $\partial_t \widehat{p}^\varepsilon(t)$ as test in the second equation in (96) to obtain

$$\int_{\Omega} (\operatorname{div}^\varepsilon \partial_t \mathbf{u}^\varepsilon(t) \partial_t \widehat{p}^\varepsilon(t) + \mathbf{K}^\varepsilon \nabla^\varepsilon \widehat{p}^\varepsilon(t) \cdot \nabla^\varepsilon \partial_t \widehat{p}^\varepsilon(t)) \, dx = 0.$$

Adding the above equations yields

$$\begin{aligned} \int_{\Omega} (\mathbb{C}^\varepsilon \partial_t \mathbf{E}^\varepsilon \mathbf{u}^\varepsilon(t) \cdot \partial_t \mathbf{E} \mathbf{u}^\varepsilon(t) + \mathbf{K}^\varepsilon \nabla^\varepsilon \widehat{p}^\varepsilon(t) \cdot \partial_t \nabla^\varepsilon \widehat{p}^\varepsilon(t)) \, dx \\ = \int_{\Omega} (\partial_t \bar{\mathbf{f}}^\varepsilon(t) \cdot \partial_t \mathbf{u}^\varepsilon(t) + \partial_t \widetilde{p}_{a,\varepsilon}(t) \partial_t \operatorname{div}^\varepsilon \mathbf{u}^\varepsilon(t)) \, dx. \end{aligned}$$

which holds for a.e. $t \in (0, T)$. From the above equation we obtain the following estimate

$$\|\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon\|_{H^1(0,T;L^2)}^2 + \|\nabla^\varepsilon \widehat{p}^\varepsilon\|_{L^\infty(0,T;L^2)}^2 \leq C, \quad (112)$$

whence (110c). Using (112), Korn's inequality, Poincaré inequality, and the boundary conditions we obtain that

$$\begin{aligned} \|\mathbf{u}^\varepsilon\|_{H^1(0,T;H^1)} &\leq C, \\ \|\widehat{p}^\varepsilon\|_{L^\infty(0,T;H^1)} &\leq C. \end{aligned}$$

These inequalities imply (110a,b) and

$$\widehat{p}^\varepsilon \xrightarrow{*} \widehat{p} \quad \text{in } L^\infty(0, T; H^1(\Omega)),$$

with $\widehat{p} = 0$ on $\Gamma_{p,D}$. Since $\|\partial_3 \widehat{p}^\varepsilon\|_{L^\infty(0,T;L^2)} \leq C\varepsilon$ by (112), we conclude that $\partial_3 \widehat{p} = 0$ and hence $\widehat{p} = 0$. \square

As a preliminary step, we need the following result.

Proposition 6 (Characterization of η). *The limit η is given by*

$$\eta = \frac{\tilde{p}_a - \mathbb{C}_{33\alpha\beta} E_{\alpha\beta} \mathbf{u}}{\mathbb{C}_{3333}}. \quad (113)$$

Proof. In the first equation of (96) we take as test function $\mathbf{v} = \varepsilon^2 \varphi \mathbf{e}_3$ with $\varphi(x) = \int_0^{x_3} \psi(x_1, x_2, z) dz$ where $\psi \in C_0^\infty(\Omega)$. On letting ε tend to 0, we obtain, for almost all times,

$$\int_{\Omega} (\mathbb{C}_{3333} \eta + \mathbb{C}_{33\gamma\delta} E_{\gamma\delta} \mathbf{u} - \tilde{p}_a) \psi \, dx = 0.$$

By the arbitrariness of ψ we obtain (113). \square

Remark 2. In the special case when \mathbb{C} is isotropic, we have

$$\mathbb{C}\mathbf{E} = 2G\mathbf{E} + (K - \frac{2}{3}G)\text{tr}(\mathbf{E})\mathbf{I}, \quad (114)$$

and (113) becomes

$$E_{33} = \frac{\tilde{p}_a - (K - \frac{2}{3}G) E_{\gamma\gamma}}{\frac{4}{3}G + K}$$

Theorem 2. Let \mathbf{u}^ε and \hat{p}^ε the solution of (96). As ε tends to 0,

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } H^1(0, T; H^1(\Omega; \mathbb{R}^3)), \quad (115a)$$

$$\hat{p}^\varepsilon \xrightarrow{*} 0 \quad \text{in } L^\infty(0, T; H^1(\Omega)). \quad (115b)$$

Moreover, the weak limit \mathbf{u} satisfies, for all $t \in [0, T)$, the following variational equation

$$\begin{cases} \mathbf{u} \in H^1(0, T; H_{KL}^1(\Omega)), \\ \int_{\Omega} (\bar{\mathbb{C}}_{\alpha\beta\gamma\delta} E_{\alpha\beta} \mathbf{u}(t) - D_{\gamma\delta} \tilde{p}_a(t)) E_{\gamma\delta} \mathbf{v} dx = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} dx \\ \forall \mathbf{v} \in H_{KL}^1(\Omega; \mathbb{R}^3). \end{cases} \quad (116)$$

where

$$\bar{\mathbb{C}}_{\alpha\beta\gamma\delta} = \mathbb{C}_{\alpha\beta\gamma\delta} - \frac{\mathbb{C}_{\alpha\beta 33} \mathbb{C}_{33\gamma\delta}}{\mathbb{C}_{3333}}, \quad D_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\mathbb{C}_{33\alpha\beta}}{\mathbb{C}_{3333}}. \quad (117)$$

Proof. In the weak formulation (96) we take $\mathbf{v}(x, t) = \varphi(t) \bar{\mathbf{v}}(x)$, with $\bar{\mathbf{v}} \in H_{KL}^1(\Omega; \mathbb{R}^3)$. We have $E_{3i} \bar{\mathbf{v}} = 0$, and hence

$$\mathbb{C}^\varepsilon \mathbf{E}^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{E}^\varepsilon \bar{\mathbf{v}} = \mathbb{C}_{ij\gamma\delta}^\varepsilon E_{ij}^\varepsilon \mathbf{u}^\varepsilon E_{\gamma\delta} \bar{\mathbf{v}} = \mathbb{C}_{\alpha\beta\gamma\delta}^\varepsilon E_{\alpha\beta}^\varepsilon \mathbf{u}^\varepsilon E_{\gamma\delta} \bar{\mathbf{v}} + \mathbb{C}_{33\gamma\delta}^\varepsilon E_{33}^\varepsilon \mathbf{u}^\varepsilon E_{\gamma\delta} \bar{\mathbf{v}},$$

where the second equality follows from (85e). Hence, the first equation in (96) takes the form

$$\begin{aligned} & \int_0^T \varphi(t) \int_{\Omega} (\mathbb{C}_{\alpha\beta\gamma\delta}^\varepsilon E_{\alpha\beta}^\varepsilon \mathbf{u}^\varepsilon(t) E_{\gamma\delta} \bar{\mathbf{v}} + \mathbb{C}_{33\gamma\delta}^\varepsilon E_{33}^\varepsilon \mathbf{u}^\varepsilon(t) E_{\gamma\delta} \bar{\mathbf{v}} - \hat{p}^\varepsilon(t) \bar{v}_{\alpha,\alpha}) dx dt \\ &= \int_0^T \varphi(t) \int_{\Omega} (\mathbf{f}^\varepsilon(t) \cdot \bar{\mathbf{v}} + \tilde{p}_a^\varepsilon(t) \partial_\alpha \bar{v}_\alpha) dx dt. \end{aligned} \quad (118)$$

Thus, thanks to Lemma 3, Proposition 5, and to assumption (98a) we can pass to the limit in (118) to get

$$\begin{aligned} & \int_0^T \varphi(t) \int_{\Omega} (\mathbb{C}_{\alpha\beta\gamma\delta} E_{\alpha\beta} \mathbf{u}(t) E_{\gamma\delta} \bar{\mathbf{v}} + \mathbb{C}_{33\gamma\delta} \eta(t) E_{\gamma\delta} \bar{\mathbf{v}}) \, dx \, dt \\ &= \int_0^T \varphi(t) \int_{\Omega} (\mathbf{f}(t) \cdot \bar{\mathbf{v}} + \tilde{p}_a(t) \bar{v}_{\alpha,\alpha}) \, dx \, dt. \end{aligned}$$

By the arbitrariness of φ , we have

$$\int_{\Omega} (\mathbb{C}_{\alpha\beta\gamma\delta} E_{\alpha\beta} \mathbf{u}(t) E_{\gamma\delta} \bar{\mathbf{v}} + \mathbb{C}_{33\gamma\delta} \eta(t) E_{\gamma\delta} \bar{\mathbf{v}} - \tilde{p}_a(t) \bar{v}_{\alpha,\alpha}) \, dx = \int_{\Omega} \mathbf{f}(t) \cdot \bar{\mathbf{v}} \, dx,$$

which holds for all $t \in [0, T)$ by continuity. Now we use Proposition 6 to express η in terms of $E_{\alpha\beta} \mathbf{u}$ and \tilde{p}_a to obtain the thesis. \square

The plate equations. As a first step towards the deduction of plate equations is the following representation result, whose proof may be found, for instance, in [11, Thm. 1.4-1.(c)]).

Proposition 7 (Characterization of $H_{KL}^1(\Omega; \mathbb{R}^3)$). *A displacement $\mathbf{v} : \Omega \rightarrow \mathbb{R}^3$ is an element of $H_{KL}^1(\Omega; \mathbb{R}^3)$ if and only if there exist $z_{\alpha} \in H^1(\omega)$ and $z_3 \in H^2(\omega)$ such that $z_i = 0$ on $\gamma_{u,D}$, $\partial_n z_3 = 0$ on $\gamma_{u,D}$ and*

$$v_{\alpha}(x) = z_{\alpha}(x_1, x_2) - x_3 z_{3,\alpha}(x_1, x_2), \quad v_3(x_1, x_2) = z_3(x_1, x_2) \quad (119)$$

for a.e. $x \in \Omega$.

According to Proposition 7 there exist

$$w_{\alpha} \in H^1(0, T; H^1(\omega)), \quad w_3 \in H^1(0, T; H^2(\omega)) \quad (120)$$

such that the limit displacement field obtained in Theorem 2 admits the representation

$$u_{\alpha}(x, t) = w_{\alpha}(x_1, x_2, t) - x_3 \partial_{\alpha} w_3(x_1, x_2, t), \quad u_3(x, t) = w_3(x_1, x_2, t) \quad (121)$$

for a.e. $(x, t) \in \Omega \times (0, T)$. Thus,

$$E_{\alpha\beta} \mathbf{u} = E_{\alpha\beta} \mathbf{w} - x_3 \partial_{\alpha\beta} w_3. \quad (122)$$

As a second step, we turn to the variational statement in (116) that characterizes the limit \mathbf{u} , namely,

$$\begin{aligned} \int_{\Omega} S_{\alpha\beta}(t) E_{\gamma\delta} \mathbf{v} dx &= \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} dx \\ \forall \mathbf{v} \in H_{KL}^1(\Omega) : \mathbf{v} &= \mathbf{0} \text{ on } \Gamma_{u,D}, \end{aligned} \quad (123)$$

with

$$S_{\alpha\beta} := \overline{\mathbb{C}}_{\alpha\beta\gamma\delta} E_{\gamma\delta} \mathbf{u} - D_{\alpha\beta} \tilde{p}_a, \quad (124)$$

a statement that holds at all times $t \in [0, T)$. We substitute into (123) the representation (119) and (121) of the unknown \mathbf{u} and the test \mathbf{v} . The result is a pair of variational statements involving the unknowns \mathbf{w} and the test functions \mathbf{z} . The first statement involves the in-plane components z_{α} of the test function \mathbf{z} , and has the form

$$\begin{aligned} \int_{\omega} N_{\alpha\beta}(t) E_{\alpha\beta} \mathbf{z} dx_1 dx_2 &= \int_{\omega} r_{\alpha}(t) z_{\alpha} dx_1 dx_2 \\ \forall \mathbf{z} = (z_1, z_2) \in H^1(\omega; \mathbb{R}^2) : \mathbf{z} &= \mathbf{0} \text{ on } \gamma_{u,D}, \end{aligned} \quad (125)$$

with

$$N_{\alpha\beta}(x_1, x_2, t) := \int_{-h/2}^{+h/2} S_{\alpha\beta}(x, t) dx_3, \quad (126)$$

and

$$r_{\alpha}(x_1, x_2, t) := \int_{-h/2}^{+h/2} f_{\alpha}(x, t) dx_3; \quad (127)$$

the second statement is deduced by considering the transversal component z_3 of the test \mathbf{z} , and reads

$$\begin{aligned} \int_{\omega} M_{\alpha\beta}(t) (-\partial_{\alpha\beta} z_3) dx_1 dx_2 &= \int_{\omega} s_{\alpha}(t) (-\partial_{\alpha} z_3) dx_1 dx_2 \\ \forall z_3 \in H^2(\omega) : z_3 &= 0 \text{ and } \partial_n z_3 = 0 \text{ on } \gamma_{u,D}, \end{aligned} \quad (128)$$

where

$$M_{\alpha\beta}(x_1, x_2, t) := \int_{-h/2}^{+h/2} x_3 S_{\alpha\beta}(x, t) dx_3, \quad (129)$$

and

$$s_{\alpha}(x_1, x_2, t) := \int_{-h/2}^{+h/2} x_3 f_{\alpha}(x, t) dx_3. \quad (130)$$

On introducing the shorthand notation:

$$\varphi^{(i)}(x_1, x_2, t) = \frac{(i+1)2^i}{h} \int_{-h/2}^{h/2} \left(\frac{x_3}{h}\right)^i \varphi(x, t) dx_3. \quad (131)$$

and on combining (122) and (124), and (99) with (126) we can write the explicit expression of the tension forces $N_{\alpha\beta}$ as

$$N_{\alpha\beta} = h\overline{\mathbb{C}}_{\alpha\beta\gamma\delta}^{(0)}E_{\gamma\delta}\mathbf{w} + \frac{h^2}{4}\overline{\mathbb{C}}_{\alpha\beta\gamma\delta}^{(1)}(-\partial_{\gamma\delta}w_3) - hD_{\alpha\beta}^{(0)}\frac{p^+ + p^-}{2} - (\zeta D_{\alpha\beta})^{(0)}(p^+ - p^-); \quad (132)$$

likewise, combination with (129) yields

$$M_{\alpha\beta} = \frac{h^3}{12}\overline{\mathbb{C}}_{\alpha\beta\gamma\delta}^{(2)}(-\partial_{\gamma\delta}w_3) + \frac{h^2}{4}\overline{\mathbb{C}}_{\alpha\beta\gamma\delta}^{(1)}E_{\gamma\delta}\mathbf{w} - \frac{h^2}{4}D_{\alpha\beta}^{(1)}\frac{p^+ + p^-}{2} - \frac{h}{4}(\zeta D_{\alpha\beta})^{(1)}(p^+ - p^-). \quad (133)$$

Isotropic material response independent on x_3 . We now specialize our results to a situation when the tensors \mathbb{C} and \mathbf{K} are isotropic and do not depend on the coordinate x_3 . In this case, we have $\mathbb{C}_{ijkl} = 2G\delta_{ik}\delta_{jl} + \lambda\delta_{ij}\delta_{kl}$ and $K_{ij} = \kappa\delta_{ij}$ with κ a positive constant. An elementary calculation yields

$$\overline{\mathbb{C}}_{\alpha\beta\gamma\delta} = 2G\delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{2G}{2G + \lambda}\lambda\delta_{\alpha\beta}\delta_{\gamma\delta}, \quad D_{\alpha\beta} = \delta_{\alpha\beta}. \quad (134)$$

In this case, we have $\overline{\mathbb{C}}_{\alpha\beta\gamma\delta}^{(i)} = \overline{\mathbb{C}}_{\alpha\beta\gamma\delta}$ if i is even and $\overline{\mathbb{C}}_{\alpha\beta\gamma\delta}^{(i)} = 0$ if i is odd. Likewise, $D_{\alpha\beta}^{(i)} = \delta_{\alpha\beta}^{(i)} = \delta_{\alpha\beta}$ if i is even and $D_{\alpha\beta}^{(i)} = 0$ if i is odd. Moreover, as observed in Remark 1, the independence of \mathbf{K} on x_3 entails that $\zeta(x) = x_3$ in the representation formula (99); thus, $(\zeta D_{\alpha\beta})^{(i)} = (x_3\delta_{\alpha\beta})^{(i)} = h(i+1)/(2(i+2))\delta_{\alpha\beta}$ if i is odd and $(\zeta D_{\alpha\beta})^{(i)} = 0$ if i is even. As a result, (132) and (133) become, respectively,

$$N_{\alpha\beta} = 2GhE_{\alpha\beta}\mathbf{w} + \frac{2G\lambda h}{2G + \lambda}(\partial_{\gamma}w_{\gamma})\delta_{\alpha\beta} - h\frac{p^+ + p^-}{2}\delta_{\alpha\beta}, \quad (135)$$

and

$$M_{\alpha\beta} = \frac{h^3}{12}2G(-\partial_{\alpha\beta}w_3) - \frac{h^3}{12}\frac{2G\lambda}{2G + \lambda}\Delta w_3\delta_{\alpha\beta} - \frac{h^2}{12}(p^+ - p^-)\delta_{\alpha\beta}. \quad (136)$$

6 A diffusive plate model

In this section we consider a plate with mobility through the thickness of the plate much smaller than that in the plane, that is $K_{33}^{\varepsilon} \ll K_{\alpha\beta}^{\varepsilon}$. A similar assumption was also made in [22]. In particular, we assume that \mathbf{K}^{ε} has the following scaling

$$\mathbf{K}^{\varepsilon} := \begin{pmatrix} K_{\alpha\beta} & 0 \\ 0 & \varepsilon^2 K_{33} \end{pmatrix}$$

where for simplicity we have taken $K_{a3}^\varepsilon = 0$ and where $K_{\alpha\beta}$ and K_{33} are functions whose regularity is specified below. Under this position, for any functions f and g we have that $\mathbf{K}^\varepsilon \nabla^\varepsilon f \cdot \nabla^\varepsilon g = \mathbf{K} \nabla f \cdot \nabla g$ where

$$\mathbf{K} := \begin{pmatrix} K_{\alpha\beta} & 0 \\ 0 & K_{33} \end{pmatrix}.$$

Within this setting, Problem (95) rewrites as

$$\forall t \in [0, T] : \begin{cases} \tilde{p}_a^\varepsilon \in H^1(\Omega), & \tilde{p}_a^\varepsilon(t) = p_a^\varepsilon(t) \text{ on } \Gamma_{p,D}, \\ \int_{\Omega} \mathbf{K} \nabla \tilde{p}_a^\varepsilon(t) \cdot \nabla q \, dx = 0 & \forall q \in H_{p,D}^1(\Omega), \end{cases} \quad (137)$$

and Problem (96) as

$$\begin{cases} \mathbf{u}^\varepsilon \in H^1(0, T; H_{u,D}^1(\Omega; \mathbb{R}^3)), \\ \bar{p}^\varepsilon \in L^\infty(0, T; H_{p,D}^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \text{with} \quad \bar{p}^\varepsilon(0) = \bar{p}_0^\varepsilon, \\ \int_{\Omega} (\mathbb{C}^\varepsilon \mathbf{E}^\varepsilon \mathbf{u}^\varepsilon(t) \cdot \mathbf{E}^\varepsilon \mathbf{v} - \bar{p}^\varepsilon(t) \operatorname{div}^\varepsilon \mathbf{v}) \, dx = \int_{\Omega} (\mathbf{f}^\varepsilon(t) \cdot \mathbf{v} + \tilde{p}_a^\varepsilon \operatorname{div}^\varepsilon \mathbf{v}) \, dx \\ \quad \forall \mathbf{v} \in H_{u,D}^1(\Omega; \mathbb{R}^3), \forall t \in [0, T], \\ \int_{\Omega} \operatorname{div}^\varepsilon \dot{\mathbf{u}}^\varepsilon(t) q + \mathbf{K} \nabla \bar{p}^\varepsilon(t) \cdot \nabla q \, dx = 0 \quad \forall q \in H_{p,D}^1(\Omega), \text{ for a.e. } t \in (0, T). \end{cases} \quad (138)$$

We keep all the assumption stated in the previous section except (85h), (85i), and (97), that we replace by

$$\mathbf{K} \in L^\infty(\Omega; \mathbb{R}_{\text{Sym}}^{3 \times 3}), \quad (139a)$$

$$\exists c_K > 0 : \mathbf{K} \mathbf{a} \cdot \mathbf{a} \geq c_K |\mathbf{a}|^2 \quad \forall \mathbf{a} \in \mathbb{R}^3 \quad \text{a.e. in } \Omega. \quad (139b)$$

The asymptotic behaviour of problems (137) and (138) can be easily studied by following the same steps taken in the previous section. In particular, Lemma 3 will be replaced by the following lemma.

Lemma 4. *Let $\tilde{p}_a^\varepsilon(t) \in H^1(\Omega)$ solve (137) for all $t \in [0, T]$. Then*

$$\tilde{p}_a^\varepsilon \rightharpoonup \tilde{p}_a \quad \text{in } H^1(0, T; H^1(\Omega)),$$

where \tilde{p}_a is the solution of

$$\forall t \in [0, T] : \begin{cases} \tilde{p}_a \in H^1(\Omega), & \tilde{p}_a(t) = p_a(t) \text{ on } \Gamma_{p,D}, \\ \int_{\Omega} \mathbf{K} \nabla \tilde{p}_a(t) \cdot \nabla q \, dx = 0 & \forall q \in H_{p,D}^1(\Omega). \end{cases} \quad (140)$$

The proof of the Lemma 4 coincides with that of Lemma 3. Proposition 5 still holds but with (110d) replaced by

$$\bar{p}^\varepsilon \xrightarrow{*} \bar{p} \quad \text{in } L^\infty(0, T; H^1(\Omega))$$

where $\bar{p} \in L^\infty(0, T; H_{p,D}^1(\Omega))$. Within this setting $\bar{p} \neq 0$, contrary to the previous section, because only $\nabla \bar{p}^\varepsilon$, and not $\nabla^\varepsilon \bar{p}^\varepsilon$, it is bounded in $L^\infty(0, T; L^2)$. If in the proof of Proposition 6 we take into account that $\bar{p} \neq 0$ we find

$$\eta = \frac{\tilde{p}_a + \bar{p} - \mathbb{C}_{33\alpha\beta} E_{\alpha\beta} \mathbf{u}}{\mathbb{C}_{3333}}. \quad (141)$$

Note that since $\eta, E_{\alpha\beta} \mathbf{u} \in H^1(0, T; L^2(\Omega))$ it follows that also $\bar{p} \in H^1(0, T; L^2(\Omega))$.

Theorem 3. *Let \mathbf{u}^ε and \bar{p}^ε the solution of (138). As ε tends to 0,*

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } H^1(0, T; H^1(\Omega; \mathbb{R}^3)), \quad (142a)$$

$$\bar{p}^\varepsilon \xrightarrow{*} \bar{p} \quad \text{in } L^\infty(0, T; H^1(\Omega)). \quad (142b)$$

Moreover, the weak limits \mathbf{u} and \bar{p} satisfy, for all $t \in [0, T]$, the following variational equation

$$\left\{ \begin{array}{l} \mathbf{u} \in H^1(0, T; H_{KL}^1(\Omega)), \\ \bar{p} \in L^\infty(0, T; H_{p,D}^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \text{with} \quad \bar{p}(0) = \bar{p}_0, \\ \int_{\Omega} (\bar{\mathbb{C}}_{\alpha\beta\gamma\delta} E_{\alpha\beta} \mathbf{u}(t) - D_{\gamma\delta}(\tilde{p}_a + \bar{p})(t)) E_{\gamma\delta} \mathbf{v} dx = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} dx \\ \quad \forall \mathbf{v} \in H_{KL}^1(\Omega; \mathbb{R}^3). \\ \int_{\Omega} \left(D_{\alpha\beta} E_{\alpha\beta} \dot{\mathbf{u}}(t) - \frac{\partial_t \tilde{p}_a(t) + \partial_t \bar{p}(t)}{\mathbb{C}_{3333}} \right) q + \mathbf{K} \nabla \bar{p}(t) \cdot \nabla q dx = 0 \\ \quad \forall q \in H_{p,D}^1(\Omega), \text{ for a.e. } t \in (0, T), \end{array} \right. \quad (143)$$

where $\bar{\mathbb{C}}$ and D are defined in (117).

Again, the proof of Theorem 3 is very similar to that of Theorem 2; essentially it suffices to recall that $\bar{p} \neq 0$ and that by Proposition 5 and (141) we have that

$$\operatorname{div}^\varepsilon \dot{\mathbf{u}}^\varepsilon = \operatorname{tr} E^\varepsilon \dot{\mathbf{u}}^\varepsilon \rightharpoonup E_{\alpha\alpha} \dot{\mathbf{u}} + \dot{\eta} = D_{\alpha\beta} E_{\alpha\beta} \dot{\mathbf{u}} - \frac{\partial_t \tilde{p}_a + \partial_t \bar{p}}{\mathbb{C}_{3333}},$$

where we used (141).

We decided to write Theorem 3 with the same notation used in the previous section even if we could have written it, in a more compact form, in terms of

$$p(t) := \tilde{p}_a(t) + \bar{p}(t).$$

We finally note that problem (143) is not a two dimensional problem. Indeed, while the balance equation can be written over the two dimensional domain ω , since the test function \boldsymbol{v} is a Kirchhoff-Love type of displacement, the last equation appearing in (143) cannot since the diffusion is throughout Ω .

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